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# ELEMENTARY ALGEBRA

## FOR SCHOOLS.

# ELEMENTARY ALGEBRA

## FOR SCHOOLS.

BY

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# PREFACE.

THE present work is an attempt to supply a want which we have long felt ourselves, and which we believe to be shared by many experienced teachers.

In setting before a beginner the real and perplexing difficulties of elementary Algebra, there is some fear lest first lessons should degenerate into a mere mechanical manipulation of symbols, uninteresting and uninstructive, because little understood. This well known danger led us to devote special thought to the question of order; to consider, in short, what succession of the various parts of the subject would best illustrate its bearings at an early stage; and we have finally adopted an arrangement, which—if it varies somewhat from the common use of elementary text-books—is at least based upon the experience of many years, and embodies the result of frequent consultation with our colleagues and other teachers.

As regards the earlier chapters, our order has been determined mainly by two considerations: first, a desire to introduce as early as possible the practical side of the subject and some of its most interesting applications, such as easy equations and problems; and secondly, the strong opinion that all reference to compound expressions and their resolution into factors should be postponed until the usual operations of Algebra have been exemplified in the case of simple expressions. By this course the beginner soon becomes acquainted with the ordinary algebraical processes without encountering too many of their difficulties; and he is learning at the same time something of the more attractive parts of the subject. Again, by postponing Resolution into Factors until the student has acquired some freedom and readiness in the use of symbols, we are enabled to treat this important section, and all the processes to which it gives rise, more adequately than is possible where factors are introduced and disposed of in one short early chapter.

We had originally intended to arrange the chapters in an order that might be followed without deviation from beginning to end; but in the course of the work this was found impossible without very extensive subdivision in some parts, and needless repetition in others. We have therefore marked with an asterisk all articles and examples which may conveniently be omitted by a student who is reading the subject for the first time, and we have occasionally added a note suggesting the most suitable place for a section which may have to be postponed.

To one feature of the book we wish to draw special attention. After Quadratics we have introduced two recapitulatory chapters in which are discussed certain classes of questions requiring for their solution a very ready and skilful use of factors. Such questions are undoubtedly hard, and wholly out of place at an earlier stage; yet in existing text-books their importance is hardly recognised, and they are generally introduced at a point where the ordinary student cannot use or appreciate the best methods of solving them. To these harder applications of early rules in Chapters XXVIII and XXIX, we have added several useful miscellaneous theorems illustrated by carefully selected examples. These chapters are the outcome of our own experience in preparing boys for Army and University Examinations during the last twelve years; they will be found very valuable for revision, though they also contain some new and interesting matter.

The examples are very numerous, and have been compiled with great care. They are for the most part original, though some have been selected from recent Examination Papers. In the early chapters we have given an unusual number of very easy questions, many of which will be found well suited for *vivâ voce* work. In each chapter several examples are fully worked out to illustrate the most useful applications of important rules. Also, under the belief that prevalent mistakes are not sufficiently guarded against, we have given occasional notes to caution the reader against the blunders which experience shews to be almost universal among beginners. With the help of our colleagues we have endeavoured to give none but the best methods, so that students on passing from this to a more advanced treatise may have a good foundation, and have nothing to unlearn.

We have done our best to secure accuracy in all parts of the book, but in a collection of 3500 examples it is probable that some mistakes still remain undetected: we shall be thankful to receive any corrections.

We have to acknowledge our indebtedness to many kind friends for their valuable help and advice. In the arrangement of the early chapters we have been largely influenced by the opinion of the Rev. J. M. Wilson, Head Master of Clifton College; and practical suggestions received from him have been frequently worked into the text and examples throughout the book. We have also to express our thanks for the assistance and encouragement freely given by many of our Mathematical colleagues: to the Rev. H. C. Watson and Mr F. H. Stevens of Clifton College we are especially indebted for their practical hints and able criticism; while in revising the manuscript and correcting the proof sheets their aid has been most generous and untiring.

Up to the end of Surds we hope the book will be found sufficiently complete; the remaining chapters have been introduced to meet the requirements of many Elementary Examinations, but they do not profess to give more than an easy first course of Ratio, Proportion, Variation, and the Progressions. These subjects will be treated more fully in the early chapters of our *Higher Algebra*, now in course of preparation, as a sequel to the present work.

H. S. HALL,
S. R. KNIGHT.

*June* 1885.

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## ERRATA.

p. 104, **XVI.** e. 4, *for* $9x$ *read* $\dfrac{9x}{y}$.

p. 137, **XX.** a. 26, *for* $15a^2$ *read* $45a^2$.

p. 240, *last line*, *for* $\sqrt{6}$ *read* $\sqrt[3]{6}$.

p. 254, *line* 3, *for* radical *read* radicals.

p. 273, *last line*, *for* $(a-d)+a$, *read* $(a+d)+a$.

# ALGEBRA.

## CHAPTER I.

### DEFINITIONS. SUBSTITUTIONS.

1. Algebra treats of quantities as in Arithmetic, but with greater generality; for while the quantities used in Arithmetical processes are denoted by *figures* which have one single definite value, Algebraical quantities are denoted by *symbols* which may have any value we choose to assign to them.

The symbols employed are letters, usually those of our own alphabet; and, though there is no restriction as to the numerical values a symbol may represent, it is understood that in the same piece of work it keeps the same value throughout. Thus, when we say "let $a=1$," we do not mean that $a$ must have the value 1 always, but only in the particular example we are considering. Moreover, we may operate with symbols without assigning to them any particular numerical value at all; indeed it is with such operations that Algebra is chiefly concerned.

We begin with the definitions of Algebra, premising that the usual symbols of operation $+$, $-$, $\times$, $\div$, $(\;)$, will have the same meanings as in Arithmetic.

2. An **Algebraical Expression** is a collection of symbols; it may consist of one or more **terms**, which are separated from each other by the signs $+$ and $-$. Thus $7a+5b-3c-x+2y$ is an expression consisting of five terms.

Note. When no sign precedes a term the sign $+$ is understood.

3. **Expressions** are either **simple** or **compound**. A *simple expression* consists of *one* term, as $5a$. A *compound expression*

consists of *two or more* terms. Compound expressions may be further distinguished. Thus an expression of *two* terms, as $3a - 2b$, is called a **binomial** expression; one of *three* terms, as $2a - 3b + c$, a **trinomial**; one of *more than three* terms a **multinomial**.

4. When two or more quantities are multiplied together the result is called the **product**. One important difference between the notation of Arithmetic and Algebra should be here remarked. In Arithmetic the product of 2 and 3 is written $2 \times 3$, whereas in Algebra the product of $a$ and $b$ may be written in any of the forms $a \times b$, $a . b$, or $ab$. The form $ab$ is the most usual. Thus, if $a=2$, $b=3$, the product $ab = a \times b = 2 \times 3 = 6$; but in Arithmetic 23 means "twenty-three," or $2 \times 10 + 3$.

5. Each of the quantities multiplied together to form a product is called a **factor** of the product. Thus 5, $a$, $b$ are the factors of the product $5ab$.

6. When one of the factors of an expression is a numerical quantity, it is called the **coefficient** of the remaining factors. Thus, in the expression $5ab$, 5 is the coefficient. But the word coefficient is also used in a wider sense, and it is sometimes convenient to consider any factor, or factors, of a product as the coefficient of the remaining factors. Thus, in the product $6abc$, $6a$ may be appropriately called the coefficient of $bc$. A coefficient which is not merely numerical is sometimes called a **literal** coefficient.

NOTE. When the coefficient is unity it is usually omitted. Thus we do not write $1a$, but simply $a$.

7. If a quantity be multiplied by itself any number of times, the product is called a **power** of that quantity, and is expressed by writing the number of factors to the right of the quantity and above it. Thus

$a \times a$ is called the *second power* of $a$, and is written $a^2$;
$a \times a \times a$..............*third power* of $a$, .................. $a^3$;

and so on.

The number which expresses the power of any quantity is called its **index** or **exponent**. Thus 2, 5, 7 are respectively the indices of $a^2$, $a^5$, $a^7$.

NOTE. $a^2$ is usually read "$a$ squared"; $a^3$ is read "$a$ cubed"; $a^4$ is read "$a$ to the fourth"; and so on.

When the index is unity it is omitted. Thus we do not write $a^1$, but simply $a$.

8. The beginner must be careful to distinguish between *coefficient* and *index*.

*Example* 1. If $a=4$,

$$3a=3\times a=3\times 4=12;$$

but $$a^3=a\times a\times a=4\times 4\times 4=64.$$

*Example* 2. If $b=5$,

$$4b^2=4\times b\times b=4\times 5\times 5=100;$$

whereas $$2b^4=2\times b\times b\times b\times b=2\times 5\times 5\times 5\times 5=1250.$$

*Example* 3. If $a=6$, $x=7$, then

$$\tfrac{5}{3}ax=\tfrac{5}{3}\times a\times x=\tfrac{5}{3}\times 6\times 7=70.$$

NOTE. Fractional coefficients which are greater than unity are usually kept in the form of improper fractions.

*Example* 4. If $a=4$, $x=1$ find the value of $5x^a$.

$$5x^a=5\times x^a=5\times 1^4=5\times 1=5.$$

NOTE. In the last example $1^4=1\times 1\times 1\times 1=1$. Similarly *every power* of 1 is 1.

## EXAMPLES I. a.

If $a=7$, $b=2$, $c=1$, $x=5$, $y=3$, find the value of

**1.** $14x$. **2.** $x^3$. **3.** $3ax$. **4.** $a^3$. **5.** $5by$.
**6.** $b^5$. **7.** $3b^2$. **8.** $2xa$. **9.** $6c^4$. **10.** $4y^3$.
**11.** $7c^5$. **12.** $9b^4$.

If $a=8$, $b=5$, $c=4$, $x=1$, $y=3$, find the value of

**13.** $3c^2$. **14.** $7y^3$. **15.** $5ab$. **16.** $9xy$. **17.** $8b^3$.
**18.** $3x^5$. **19.** $x^8$. **20.** $7y^4$. **21.** $c^x$. **22.** $b^y$.
**23.** $y^c$. **24.** $x^b$. **25.** $y^b$. **26.** $a^y$. **27.** $b^x$.
**28.** $a^c$.

If $a=5$, $b=1$, $c=6$, $x=4$, find the value of

**29.** $\frac{1}{4}x^4$. **30.** $\frac{5}{12}c^3$. **31.** $\frac{3}{8}x^3$. **32.** $\frac{1}{10}ax$. **33.** $3^x$.
**34.** $2^c$. **35.** $8^b$. **36.** $7^x$. **37.** $\frac{7}{15}acx$. **38.** $\frac{1}{8}bcx$.
**39.** $\frac{2}{9}c^3$. **40.** $\dfrac{x^5}{64}$.

9. When several different quantities are multiplied together a notation similar to that of Art. 7 is adopted. Thus $aabbbbcddd$ is written $a^2b^4cd^3$. And conversely $7a^3cd^2$ has the same meaning as $7\times a\times a\times a\times c\times d\times d$.

10. Each of the letters composing a term is called a **dimension** of the term, and the number of letters involved is called the **degree** of the term. Thus the product $abc$ is said to be *of three dimensions*, or *of the third degree;* and $ax^4$ is said to be *of five dimensions*, or *of the fifth degree.*

A numerical coefficient is not counted. Thus $8a^2b^5$ and $a^2b^5$ are each of *seven* dimensions.

11. But it is sometimes useful to speak of the dimensions of an expression with regard to any one of the letters it involves. For instance, the expression $8a^3b^4c$, which is of eight dimensions, may be said to be of three dimensions in $a$, of four dimensions in $b$, and of one dimension in $c$.

12. A compound expression is said to be **homogeneous** when all its terms are of the same dimensions. Thus $8a^6-a^4b^2+9ab^5$ is a *homogeneous expression of six dimensions.*

13. In dealing with Algebraical expressions, where the letters denote numerical quantities, we may make use of the principles with which the student is familiar in Arithmetic. Thus $ab$ and $ba$ each denote the product of the two quantities represented by the letters $a$ and $b$, and have therefore the same value. Again, the expressions $abc$, $acb$, $bac$, $bca$, $cab$, $cba$ have the same value, each denoting the product of the three quantities $a$, $b$, $c$. It is immaterial in what order the factors of a product are written; it is usual, however, to arrange them in alphabetical order.

*Example* 1. If $x=5$, $y=3$, find the value of $4x^2y^3$.

$$\begin{aligned}4x^2y^3&=4\times5^2\times3^3,\\&=4\times25\times27,\\&=2700.\end{aligned}$$

*Example* 2. If $a=4$, $b=9$, $x=6$, find the value of $\frac{8bx^2}{27a^3}$.

$$\begin{aligned}\frac{8bx^2}{27a^3}&=\frac{8\times9\times6^2}{27\times4^3},\\&=\frac{8\times9\times36}{27\times64},\\&=\tfrac{3}{2}.\\&=1\tfrac{1}{2}.\end{aligned}$$

14. If one factor of a product is equal to 0, the whole product must be equal to 0, *whatever values the other factors may have.* A factor 0 is sometimes called a "zero factor."

*Example* 1. If $x=0$, then $ab^3xy^2=0$, whatever be the values of $a, b, y$.

*Example* 2. If $c=0$, then $ab^2c^3=0$, whatever values $a$, $b$ may have.

## EXAMPLES I. b.

If $a=7$, $b=2$, $c=0$, $x=5$, $y=3$, find the value of

1. $4ax^2$. 2. $a^3b$. 3. $8b^2y$. 4. $3xy^2$. 5. $\frac{3}{4}b^2x$.
6. $\frac{5}{6}b^3y^2$. 7. $\frac{2}{3}xy^4$. 8. $a^3c$. 9. $a^2cy$. 10. $8x^3y$.
11. $\frac{7}{20}ab^3x$. 12. $\frac{1}{9}x^2y^4$.

If $a=2$, $b=3$, $c=1$, $p=0$, $q=4$, $r=6$, find the value of

13. $\frac{3a^2r}{8b}$. 14. $\frac{8ab^2}{9q^2}$. 15. $\frac{6a^3c}{b^2}$. 16. $\frac{4cr^2}{9a^3}$. 17. $3a^2b^c$.
18. $\frac{5}{6}ba^r$. 19. $\frac{8b^q}{9a^r}$. 20. $5a^bc^r$. 21. $\frac{2a^2p}{7r}$. 22. $3^a2^b$.
23. $2^ra^5$. 24. $c^bb^q$. 25. $\frac{5a^rb^q}{64r^a}$. 26. $\frac{27a^q}{32}$. 27. $\frac{64}{q^r}$.
28. $\frac{b^r}{r^b}$.

15. Definition. The **square root** of any proposed expression is that quantity whose square, or second power, is equal to the given expression. Thus the square root of 81 is 9, because $9^2=81$.

The square root of $a$ is denoted by $\sqrt[2]{a}$, or more simply $\sqrt{a}$.

Similarly the **cube, fourth, fifth,** &c., **root** of any expression is that quantity whose third, fourth, fifth, &c., power is equal to the given expression.

The roots are denoted by the symbols $\sqrt[3]{}$, $\sqrt[4]{}$, $\sqrt[5]{}$, &c.

*Examples.* $\sqrt[3]{27}=3$; because $3^3=27$.
$\sqrt[5]{32}=2$; because $2^5=32$.

The symbol $\sqrt{}$ is sometimes called the **radical sign.**

*Example* 1. Find the value of $5\sqrt{(6a^3b^4c)}$, when $a=3$, $b=1$, $c=8$.

$$\begin{aligned}5\sqrt{(6a^3b^4c)}&=5\times\sqrt{(6\times3^3\times1^4\times8)},\\&=5\times\sqrt{(6\times27\times8)},\\&=5\times\sqrt{1296},\\&=5\times36,\\&=180.\end{aligned}$$

*Example* 2. Find the value of $\sqrt[3]{\left(\frac{ab^4}{8x^3}\right)}$, when $a=9$, $b=3$, $x=5$.

$$\begin{aligned}\sqrt[3]{\left(\frac{ab^4}{8x^3}\right)} &= \sqrt[3]{\left(\frac{9\times 3^4}{8\times 5^3}\right)},\\ &= \sqrt[3]{\left(\frac{9\times 81}{8\times 125}\right)},\\ &= \sqrt[3]{\left(\frac{9\times 9\times 9}{1000}\right)},\\ &= \frac{9}{10}.\end{aligned}$$

## EXAMPLES I. c.

If $a=8$, $c=0$, $k=9$, $x=4$, $y=1$, find the value of

**1.** $\sqrt{(2a)}$. **2.** $\sqrt{(kx)}$. **3.** $\sqrt{(2ax)}$.

**4.** $\sqrt{(2ak^2)}$. **5.** $\sqrt[3]{(3k)}$. **6.** $\sqrt[3]{(ax^3)}$.

**7.** $\sqrt[3]{(8x^3y^3)}$. **8.** $\sqrt[3]{(cy^5)}$. **9.** $2x\sqrt{(2ay)}$.

**10.** $5y\sqrt{(4kx)}$. **11.** $3c\sqrt{(kx)}$. **12.** $2xy\sqrt{(4y^5)}$.

**13.** $\sqrt{\left(\frac{8x^3}{ak}\right)}$. **14.** $\sqrt{\left(\frac{25a}{2k}\right)}$. **15.** $\sqrt{\left(\frac{16x}{49y^3}\right)}$.

**16.** $\sqrt{\left(\frac{ca^2}{16k}\right)}$. **17.** $\sqrt[3]{\left(\frac{3a}{k^2}\right)}$. **18.** $\sqrt[3]{\left(\frac{ax^3}{27y^3}\right)}$.

**19.** $\sqrt[3]{\left(\frac{ca}{3k}\right)}$. **20.** $\sqrt[3]{\left(\frac{a^2k^2}{3x^3}\right)}$.

If $a=4$, $b=1$, $c=2$, $d=9$, $x=5$, $y=8$, find the value of

**21.** $\sqrt{(8ac)}$. **22.** $6\sqrt{(4b^3)}$. **23.** $7\sqrt{(5dx)}$.

**24.** $\sqrt{(c^5y)}$. **25.** $\sqrt{\left(\frac{1}{5xy^2}\right)}$. **26.** $\sqrt{\left(\frac{1}{8acd}\right)}$.

**27.** $\frac{1}{\sqrt{(9a^3c^2)}}$. **28.** $\frac{1}{\sqrt{(5c^4x)}}$. **29.** $\sqrt[3]{\left(\frac{1}{4cx^3}\right)}$.

**30.** $\frac{1}{\sqrt[3]{(8ab^2c)}}$. **31.** $\sqrt[3]{\left(\frac{3x^3}{d^2y}\right)}$. **32.** $\sqrt[3]{\left(\frac{5b^8}{x^4y}\right)}$.

**33.** $\sqrt{d^c}$. **34.** $\sqrt{y^a}$. **35.** $\sqrt{b^d}$. **36.** $\sqrt{d^b}$.

16. In working examples the student should pay attention to the following hints:

1. It should be clearly brought out how each step follows from the one before it; for this purpose short verbal explanations are often necessary.

2. The sign "=" should never be used except to connect quantities which are equal. Beginners should be particularly careful not to employ the sign of equality in any vague and inexact sense.

3. Unless the expressions are very short the signs of equality in the several steps of the work should be placed one under the other.

4. In elementary work too much importance cannot be attached to neatness of style and arrangement. The beginner should remember that neatness is in itself conducive to accuracy.

*Example* 1. Find the value of $\frac{3}{10}ab - 7x^2 - \frac{9}{4}ay^2 + 2b^3$, when

$$a=5,\ b=4,\ x=3,\ y=2.$$

$$\begin{aligned}\tfrac{3}{10}ab - 7x^2 - \tfrac{9}{4}ay^2 + 2b^3 &= \tfrac{3}{10} \,.\, 5 \,.\, 4 - 7 \,.\, 3^2 - \tfrac{9}{4} \,.\, 5 \,.\, 2^2 + 2 \,.\, 4^3, \\ &= 6 - 63 - 45 + 128, \\ &= 26.\end{aligned}$$

*Example* 2. Find the value of $\frac{3}{5}x^2 - a^2y + 7abx - \frac{5}{2}y^3$, when

$$a=5,\ b=0,\ x=7,\ y=1.$$

$$\begin{aligned}\tfrac{3}{5}x^2 - a^2y + 7abx - \tfrac{5}{2}y^3 &= \tfrac{3}{5} \,.\, 7^2 - 5^2 \,.\, 1 + 0 - \tfrac{5}{2} \,.\, 1^3, \\ &= 29\tfrac{2}{5} - 25 - 2\tfrac{1}{2}. \\ &= 1\tfrac{9}{10}.\end{aligned}$$

NOTE. In the last example the *zero term* does not affect the result.

*Example* 3. When $p=9$, $r=6$, $k=4$, find the value of

$$\frac{1}{3}\sqrt[3]{\left(\frac{pr}{k^2}\right)} + \sqrt{(5p+3r+1)} - \frac{2r^2}{9k}.$$

$$\begin{aligned}\frac{1}{3}\sqrt[3]{\left(\frac{pr}{k^2}\right)} + \sqrt{(5p+3r+1)} - \frac{2r^2}{9k} &= \frac{1}{3}\sqrt[3]{\left(\frac{54}{16}\right)} + \sqrt{(45+18+1)} - \frac{2\times 36}{9\times 4}. \\ &= \tfrac{1}{3}\sqrt[3]{\tfrac{27}{8}} + \sqrt{64} - 2, \\ &= \tfrac{1}{3}\times\tfrac{3}{2} + 8 - 2, \\ &= 6\tfrac{1}{2}.\end{aligned}$$

## EXAMPLES I. d.

If $a=2$, $b=3$, $c=1$, $d=0$, find the numerical value of

**1.** $6a+5b-8c+9d$.

**2.** $3a-4b+6c+5d$.

**3.** $5a+3c-2b+6d$.

**4.** $ab+bc+ca-da$.

**5.** $6ab-3cd+2da-5cb+2db$.

**6.** $abc+bcd+cda+dab$.

**7.** $3abc-2bcd+2cda-4dab$.

**8.** $2bc+3cd-4da+5ab$.

**9.** $3bcd+5cda-7dab+abc$.

**10.** $a^2+b^2+c^2+d^2$.

**11.** $2a^2+3b^3-4c^4$.

**12.** $a^4+b^4-c^4$.

If $a=1$, $b=2$, $c=3$, $d=0$, find the numerical value of

**13.** $a^3+b^3+c^3+d^3$.

**14.** $\frac{1}{2}bc^3-a^3-b^3-\frac{3}{4}ab^3c$.

**15.** $3abc-b^2c-6a^3$.

**16.** $2a^2+2b^2+2c^2+2d^2-2bc-2cd-2da-2ab$.

**17.** $c^3+\frac{4}{5}ad^4-3a^3+b^2d$.

**18.** $a^2+2b^2+2c^2+d^2+2ab+2bc+\frac{2}{7}cd$.

**19.** $2c^2+2a^2+2b^2-4cb+6abcd$.

**20.** $13a^2+\frac{11}{9}c^4+20ab-16ac-16bc$.

**21.** $6ab-\frac{4}{3}ac^2-2a+\frac{1}{8}b^4-3d+\frac{4}{9}c^3$.

**22.** $a^2-c^2+b^2-d^2+2ab-2cd$.

**23.** $2ab-\frac{3}{4}b^3+3ac-2c-d+\frac{4}{15}ad$.

**24.** $125b^4c-9d^5+3abc^2d$.

If $a=8$, $b=6$, $c=1$, $x=9$, $y=4$, find the value of

**25.** $\frac{5}{3}a-\frac{1}{9}b^3+\frac{7}{8}y^2$.

**26.** $\frac{5}{27}ax-\dfrac{32}{y^2}-\dfrac{6a}{cxy}$.

**27.** $\dfrac{3a^2b}{cxy^2}-\dfrac{5y}{a}$.

**28.** $\sqrt[3]{\left(\dfrac{6cy^4}{x^2}\right)}+2\sqrt{\left(\dfrac{3a^3}{4b^3}\right)}$.

**29.** $\frac{2}{9}ab^2-\sqrt{\dfrac{x}{c^3}}-bc^3y$.

**30.** $\sqrt[3]{(bxy)}-\frac{1}{8}b^2+\dfrac{8x^2}{b^2y}$.

**31.** $\frac{3}{4}ac-\sqrt{\left(\dfrac{b^2}{9y}\right)}-\sqrt[3]{\left(\dfrac{by}{x^2}\right)}$.

**32.** $\dfrac{5b^2y^3}{12a^2x}-\sqrt[3]{\left(\dfrac{x^4a}{b^2y^2}\right)}+\sqrt{\left(\dfrac{ab^3}{3x}\right)}$.

# CHAPTER II.

## NEGATIVE QUANTITIES. ADDITION OF LIKE TERMS.

17. IN his Arithmetical work the student has been accustomed to deal with numerical quantities connected by the signs + and − ; and in finding the value of an expression such as $1\frac{3}{4}+7\frac{2}{3}-3\frac{1}{8}+6-4\frac{1}{3}$ he understands that the quantities to which the sign + is prefixed are *additive*, and those to which the sign − is prefixed are *subtractive*, while the first quantity, $1\frac{3}{4}$, to which no sign is prefixed, is counted among the additive terms. The same notions prevail in Algebra; thus in using the expression $7a+3b-4c-2d$ we understand the symbols $7a$ and $3b$ to be additive, while $4c$ and $2d$ are subtractive.

18. But in Arithmetic the sum of the additive terms is always greater than the sum of the subtractive terms; and if the reverse were the case the result would have no Arithmetical meaning. In Algebra, however, not only may the sum of the subtractive terms exceed that of the additive, but a subtractive term may stand alone, and yet have a meaning quite intelligible.

And so it is usual to divide all Algebraical quantities into **positive quantities** and **negative quantities**, according as they are expressed with the sign + or the sign −; and this is quite irrespective of any actual process of addition and subtraction.

This idea may be made clearer by one or two simple illustrations.

I. Suppose a man were to gain £100 and then lose £70, his total *gain* would be £30. But if he first gains £70 and then loses £100 the result of his trading is a *loss* of £30.

The corresponding Algebraical statements would be

$$£100 - £70 \;= +£30,$$

$$£70 \;- £100 = -£30,$$

and the negative quantity in the second case is interpreted as a *debt*, that is, a sum of money opposite in character to the positive quantity, or *gain*, in the first case; in fact it may be said to possess a subtractive quality which would produce its effect on a *future* transaction, or perhaps wholly counteract a *future* gain.

II. Suppose a man were to walk along a straight road 100 yards forwards and then 70 yards backwards, his distance from the starting-point would be 30 yards. But if he first walks 70 yards forwards and then 100 yards backwards his distance from the starting-point would be 30 yards, but *on the opposite side of it.* As before we have

$$100 \text{ yards} - \ 70 \text{ yards} = +30 \text{ yards},$$
$$70 \text{ yards} - 100 \text{ yards} = -30 \text{ yards},$$

and here we see that the negative sign may be taken as indicating *a reversal of direction.*

Many other illustrations might be chosen; but it will be sufficient here to remind the student that some consistent interpretation can always be given, when necessary, to the negative quantities he may meet with. In applying the rules we are about to give for the addition and subtraction of algebraical quantities, it is not necessary to recur constantly to the nature and meaning of the quantities employed, whether positive or negative, though it will be found by experience that these rules will always be justified when any of the symbolical results of Algebra are interpreted into the language of common life.

19. DEFINITION. When terms do not differ, or when they differ only in their numerical coefficients, they are called **like**, otherwise they are called **unlike**. Thus $3a$, $7a$; $5a^2b$, $2a^2b$; $3a^3b^2$, $-4a^3b^2$ are pairs of like terms; and $4a$, $3b$; $7a^2$, $9a^2b$ are pairs of unlike terms.

The Rules for adding together like terms are

RULE I. *The sum of a number of like terms is a like term.*

RULE II. *If all the terms are positive, add the coefficients.*

*Example.* $8a + 9a = 17a$.

The truth of this will be recognised by the beginner when he remembers that 8 lbs. added to 9 lbs. gives 17 lbs.,

and so $\qquad 8a + 9a = 17a.$

Similarly $\qquad 8a + 9a + a + 2a + 7a = 27a.$

RULE III. *If all the terms are negative, add the coefficients numerically and prefix the minus sign to the sum.*

*Example.* The sum of $-3x$, $-5x$, $-7x$, $-x$ is $-16x$.

For a sum of money diminished successively by £3, £5, £7 and £1 is diminished altogether by £16.

RULE IV. *If the terms are not all of the same sign, add together separately the coefficients of all the positive terms and the coefficients of all the negative terms; the difference of these two results, preceded by the sign of the greater, will give the coefficient of the sum required.*

*Example* 1. The sum of $17x$ and $-8x$ is $9x$, for the difference of 17 and 8 is 9, and the greater is positive.

*Example* 2. To find the sum of $8a$, $-9a$, $-a$, $3a$, $4a$, $-11a$, $a$.

The sum of the coefficients of the positive terms is 16,

.......................................... negative ......... 21.

The difference of these is 5, and the sign of the greater is negative; hence the required sum is $-5a$.

We need not however adhere strictly to this rule, for since terms may be added or subtracted in any order, we may choose the order we find most convenient.

NOTE. The sum of two quantities numerically equal but with opposite signs is zero. Thus the sum of $5a$ and $-5a$ is 0.

*Example* 3. Find the sum of $\frac{2}{3}a$, $3a$, $-\frac{1}{6}a$, $-2a$.

The sum $= 3\frac{2}{3}a - 2\frac{1}{6}a$,

$= 1\frac{1}{2}a$,

$= \frac{3}{2}a$.

## EXAMPLES II.

Find the sum of

**1.** $5a$, $7a$, $11a$, $a$, $23a$.

**2.** $4x$, $x$, $3x$, $7x$, $9x$.

**3.** $7b$, $10b$, $11b$, $9b$, $2b$.

**4.** $6c$, $8c$, $2c$, $15c$, $19c$, $100c$, $c$.

**5.** $-3x$, $-5x$, $-11x$, $-7x$.

**6.** $-5b$, $-6b$, $-11b$, $-18b$.

**7.** $-3y$, $-7y$, $-y$, $-2y$, $-4y$.

**8.** $-c$, $-2c$, $-50c$, $-13c$.

**9.** $-11b$, $-5b$. $-3b$, $-b$.

**10.** $5x$, $-x$, $-3x$, $2x$, $-x$.

**11.** $26y$, $-11y$, $-15y$, $y$, $-3y$, $2y$.

**12.** $5f,\ -9f,\ -3f,\ 21f,\ -30f.$ **13.** $2s,\ -3s,\ s,\ -s,\ -5s,\ 5s.$

**14.** $7y,\ -11y,\ 16y,\ -3y,\ -2y.$

**15.** $5x,\ -7x,\ -2x,\ 7x,\ 2x,\ -5x.$

**16.** $7ab,\ -3ab,\ -5ab,\ 2ab,\ ab.$

Find the value of

**17.** $-9x^2+11x^2+3x^2-4x^2.$ **18.** $3a^2x-18a^2x+a^2x.$

**19.** $3a^3-7a^3-8a^3+2a^3-11a^3.$ **20.** $4x^3-5x^3-8x^3-7x^3.$

**21.** $4a^2b^2-a^2b^2-7a^2b^2+5a^2b^2-a^2b^2.$

**22.** $-9x^4-4x^4-12x^4+13x^4-7x^4.$

**23.** $7abcd-11abcd-41abcd+2abcd.$

**24.** $\frac{1}{2}x-\frac{1}{3}x+x+\frac{2}{3}x.$ **25.** $\frac{3}{2}a+\frac{3}{5}a-\frac{1}{2}a.$

**26.** $-5b+\frac{1}{4}b-\frac{3}{2}b+2b-\frac{1}{2}b+\frac{7}{4}b.$

**27.** $-\frac{5}{3}x^2-2x^2-\frac{2}{3}x^2+x^2+\frac{1}{2}x^2+\frac{11}{6}x^2.$

**28.** $-ab-\frac{1}{2}ab-\frac{1}{3}ab-\frac{1}{4}ab-\frac{1}{6}ab+ab+\frac{5}{12}ab.$

**29.** $\frac{2}{3}x-\frac{3}{4}x+\frac{5}{6}x-2x+\frac{11}{6}x-\frac{1}{3}x+x.$

**30.** $-\frac{5}{3}x^2-\frac{3}{4}x^2-\frac{4}{3}x^2-\frac{1}{4}x^2-x^2.$

# CHAPTER III.

## SIMPLE BRACKETS. ADDITION.

20. WHEN a number of Arithmetical quantities are connected together by the signs + and −, the value of the result is the same in whatever order the terms are taken. This also holds in the case of Algebraical quantities.

Thus $a-b+c$ is equivalent to $a+c-b$, for in the first of the two expressions $b$ is taken from $a$, and $c$ added to the result; in the second $c$ is added to $a$, and $b$ taken from the result. Similar reasoning applies to all Algebraical expressions. Hence we may write the terms of an expression in any order we please.

Thus it appears that the expression $a-b$ may be written in the equivalent form $-b+a$.

To illustrate this we may suppose, as in Art. 18, that $a$ represents a gain of $a$ pounds, and $-b$ a loss of $b$ pounds: it is clearly immaterial whether the gain precedes the loss, or the loss precedes the gain.

21. A bracket ( ) indicates that the terms enclosed within it are to be considered as one quantity. The full use of brackets will be considered in Chap. VII.; here we shall deal only with the simpler cases.

$8+(13+5)$ means that 13 and 5 are to be added and their sum added to 8. It is clear that 13 and 5 may be added separately or together without altering the result.

Thus $$8+(13+5)=8+13+5=26.$$

Similarly $a+(b+c)$ means that the sum of $b$ and $c$ is to be added to $a$.

Thus $$a+(b+c)=a+b+c.$$

$8+(13-5)$ means that to 8 we are to add the excess of 13 over 5; now if we add 13 to 8 we have added 5 too much, and must therefore take 5 from the result.

Thus $$8+(13-5)=8+13-5=16.$$

Similarly $a+(b-c)$ means that to $a$ we are to add $b$, diminished by $c$.

Thus $$a+(b-c)=a+b-c \text{........................} (1).$$

In like manner,
$$a+b-c+(d-e-f)=a+b-c+d-e-f \text{ .........} (2).$$

Conversely,
$$a+b-c+d-e-f=a+b-c+(d-e-f) \text{ .........} (3).$$

Again, $a-b+c=a+c-b$, [Art. 20.]
=the sum of $a$ and $c-b$,
=the sum of $a$ and $-b+c$, [Art. 20.]

therefore $$a-b+c=a+(-b+c) \text{ ..........................} (4).$$

By considering the results (1), (2), (3), (4) we are led to the following rule:

Rule. *When an expression within brackets is preceded by the sign* $+$, *the brackets can be removed without making any change in the expression.*

Conversely: *Any part of an expression may be enclosed within brackets and the sign* $+$ *prefixed, the sign of every term within the brackets remaining unaltered.*

Thus the expression $a-b+c-d+e$ may be written in any of the following ways,
$$a+(-b+c-d+e),$$
$$a-b+(c-d+e),$$
$$a-b+c+(-d+e).$$

22. The expression $a-(b+c)$ means that from $a$ we are to take the sum of $b$ and $c$. The result will be the same whether $b$ and $c$ are subtracted separately or in one sum. Thus
$$a-(b+c)=a-b-c.$$

Again, $a-(b-c)$ means that from $a$ we are to subtract the excess of $b$ over $c$. If from $a$ we take $b$ we get $a-b$; but by so doing we shall have taken away $c$ too much, and must therefore add $c$ to $a-b$. Thus
$$a-(b-c)=a-b+c.$$

In like manner,
$$a-b-(c-d-e)=a-b-c+d+e.$$

Accordingly the following rule may be enunciated:

RULE. *When an expression within brackets is preceded by the sign* $-$, *the brackets may be removed if the sign of every term within the brackets be changed.*

Conversely: *Any part of an expression may be enclosed within brackets and the sign* $-$ *prefixed, provided the sign of every term within the brackets be changed.*

Thus the expression $a-b+c+d-e$ may be written in any of the following ways,

$$a-(+b-c-d+e),$$
$$a-b-(-c-d+e),$$
$$a-b+c-(-d+e).$$

## ADDITION.

23. When two or more *like* terms are to be added together we have seen that they may be collected and the result expressed as a single like term. If, however, the terms are *unlike* they cannot be collected. Thus we write the sum of $a$ and $b$ in the form $a+b$.

Also, by the rules for removing brackets, $a+(-b)=a-b$; that is, the sum of $a$ and $-b$ is written in the form $a-b$.

*Example* 1. Find the sum of $x$, $x^2$, $x^3$.

Since different powers of the same letter are unlike terms, the sum is $x+x^2+x^3$.

This expression cannot be abridged.

*Example* 2. The sum of $x$, $-x^2$, $-x^3$, $x^4$ is $x-x^2-x^3+x^4$.

*Example* 3. Find the sum of $3a-5b+2c$ and $2a+3b-c$.

The result may be written $3a-5b+2c+(2a+3b-c)$,

$$=3a-5b+2c+2a+3b-c,$$
$$=3a+2a-5b+3b+2c-c,$$
$$=5a-2b+c,$$

by collecting the like terms.

The operation is more conveniently performed as follows:

$$\begin{array}{r} 3a-5b+2c \\ 2a+3b-\phantom{2}c \\ \hline 5a-2b+\phantom{2}c. \end{array}$$

RULE. *Arrange the expressions in lines so that the like terms may be in the same vertical columns: then add each column beginning with that on the left.*

*Example.* Find the sum of $3a-7b+2c$; $6a-b+5c$; $-4a+3b-8c$.

$$\begin{array}{r} 3a-7b+2c \\ 6a-\phantom{7}b+5c \\ -4a+3b-8c \\ \hline 5a-5b-\phantom{2}c. \end{array}$$

24. From the foregoing Examples it will be observed that in Algebra the word *sum* is used in a wider sense than in Arithmetic. Thus, in the language of Arithmetic, $a-b$ signifies that $b$ is to be subtracted from $a$, and bears that meaning only; but in Algebra it is also taken to mean the sum of the two quantities $a$ and $-b$ without any regard to the relative magnitudes of $a$ and $b$.

When quantities are connected by the signs + and −, the resulting expression is called their **Algebraical Sum.**

Thus $$11a-27a+13a=-3a$$
states that the Algebraical Sum of $11a$, $-27a$, and $13a$ is equal to $-3a$.

## EXAMPLES III. a.

Find the sum of

1. $a+2b-3c$; $-3a+b+2c$; $2a-3b+c$.
2. $3a+2b-c$; $-a+3b+2c$; $2a-b+3c$.
3. $-3x+2y+z$; $x-3y+2z$; $2x+y-3z$.
4. $-x+2y+3z$; $3x-y+2z$; $2x+3y-z$.
5. $4a+3b+5c$; $-2a+3b-8c$; $a-b+c$.
6. $-15a-19b-18c$; $14a+15b+8c$; $a+5b+9c$.
7. $25a-15b+c$; $13a-10b+4c$; $a+20b-c$.
8. $-16a-10b+5c$; $10a+5b+c$; $6a+5b-c$.
9. $5ax-7by+cz$; $ax+2by-cz$; $-3ax+2by+3cz$.
10. $20p+q-r$; $p-20q+r$; $p+q-20r$.

Add together the following expressions:

11. $-5ab+6bc-7ca$; $8ab-4bc+3ca$; $-2ab-2bc+4ca$.
12. $15ab-27bc-6ca$; $14ab-18bc+10ca$; $-49ab+45bc-3ca$.
13. $5ab+bc-3ca$; $ab-bc+ca$; $-ab+bc+2ca$.
14. $pq+qr-rp$; $-pq+qr+rp$; $pq-qr+rp$.
15. $x+y+z$; $2x+3y-2z$; $3x-4y+z$.

**16.** $2a-3b+c$; $15a-21b-8c$; $3a+24b+7c$.

**17.** $4xy-9yz+2zx$; $-25xy+24yz-zx$; $23xy-15yz+zx$.

**18.** $17ab-13bc+8ca$; $-5ab+9bc-7ca$; $2ab-7bc-ca$.

**19.** $47x-63y+z$; $-25x+15y-3z$; $-22x+48y+15z$.

**20.** $23a-17b-2c$; $-9a+15b+7c$; $-13a+3b-4c$.

25. In adding together several algebraical expressions containing terms with different powers of the same letter, it will be found convenient to arrange all the expressions in *descending* or *ascending* powers of that letter.

*Example* 1. Add together $3x^3+7-5x^2$; $2x^2-8-9x$; $4x-2x^3+3x^2$.

$$\begin{array}{rrrr}
3x^3 & -5x^2 & & +7 \\
 & 2x^2 & -9x & -8 \\
-2x^3 & +3x^2 & +4x & \\
\hline
x^3 & & -5x & -1.
\end{array}$$

The result is in *descending* powers of $x$.

*Example* 2. Add together

$3ab^2-2b^3+a^3$; $5a^2b-ab^2-3a^3$; $8a^3+5b^3$; $9a^2b-2a^3+ab^2$.

$$\begin{array}{rrrr}
-2b^3 & +3ab^2 & & +a^3 \\
 & -ab^2 & +5a^2b & -3a^3 \\
5b^3 & & & +8a^3 \\
 & ab^2 & +9a^2b & -2a^3 \\
\hline
3b^3 & +3ab^2 & +14a^2b & +4a^3.
\end{array}$$

In this example the student should notice that the result is expressed according to *descending* powers of $b$, and *ascending* powers of $a$.

## EXAMPLES III. b.

Find the sum of

**1.** $2ab+3ca+6abc$; $-5ab+2bc-5abc$; $3ab-2bc-3ca$.

**2.** $2x^2-2xy+3y^2$; $4y^2+5xy-2x^2$; $x^2-2xy-6y^2$.

**3.** $3a^2-7ab-4b^2$; $-6a^2+9ab-3b^2$; $4a^2+ab+5b^2$.

**4.** $x^2+xy-y^2$; $-z^2+yz+y^2$; $-x^2+xz+z^2$.

**5.** $-x^2-3xy+3y^2$; $3x^2+4xy-5y^2$; $x^2+xy+y^2$.

6. $x^3-x^2+x-1$; $2x^2-2x+2$; $-3x^3+5x+1$.

7. $2x^3-x^2-x$; $4x^3+8x^2+7x$; $-6x^3-6x^2+x$.

8. $9x^2-7x+5$; $-14x^2+15x-6$; $20x^2-40x-17$.

9. $10x^3+5x+8$; $3x^3-4x^2-6$; $2x^3-2x-3$.

10. $a^3-ab+bc$; $ab+b^3-ca$; $ca-bc+c^3$.

11. $5a^3-3c^3+d^3$; $b^3-2a^3+3d^3$; $4c^3-2a^3-3d^3$.

12. $6x^3-2x+1$; $2x^3+x+6$; $x^2-7x^3+2x-4$.

13. $a^3-a^2+3a$; $3a^3+4a^2+8a$; $5a^3-6a^2-11a$.

14. $x^2+y^2-2xy$; $2z^2-3y^2-4yz$; $2x^2-2z^2-3xz$.

15. $x^3-2y^3+x$; $y^3-2x^3+y$; $x^2+2y^2-x+y^3$.

16. $x^3+3x^2y+3xy^2$; $-3x^2y-6xy^2-x^3$; $3x^2y+4xy^2$.

17. $a^3+5ab^2+b^3$; $b^3-10ab^2-a^3$; $5ab^2-2b^3+2a^2b$.

18. $x^5-4x^4y-5x^3y^3$; $3x^4y+2x^3y^3-6xy^4$; $3x^3y^3+6xy^4-y^5$.

19. $a^3-4a^2b+6abc$; $a^2b-10abc+c^3$; $b^3+3a^2b+abc$.

20. $x^3-4x^2y+6xy^2$; $2x^2y-3xy^2+2y^3$; $y^3+3x^2y+4xy^2$.

Add together the following expressions:

21. $\frac{1}{2}a-\frac{1}{3}b$; $-a+\frac{2}{3}b$; $\frac{3}{4}a-b$.

22. $-\frac{1}{3}a-\frac{1}{4}b$; $-\frac{2}{3}a+\frac{3}{4}b$; $-2a-b$.

23. $-2a+\frac{5}{2}c$; $-\frac{1}{3}a-2b$; $\frac{8}{3}b-3c$.

24. $-\frac{13}{8}a-\frac{11}{4}c$; $2a-3b$; $\frac{11}{5}b-c$.

25. $\frac{2}{3}x^2+\frac{1}{3}xy-\frac{1}{4}y^2$; $-x^2-\frac{2}{3}xy+2y^2$; $\frac{2}{3}x^2-xy-\frac{5}{4}y^2$.

26. $3a^2-\frac{2}{5}ab-\frac{1}{2}b^2$; $-\frac{3}{2}a^2+2ab-\frac{2}{3}b^2$; $-\frac{2}{3}a^2-ab+b^2$.

27. $\frac{5}{8}x^2-\frac{1}{3}xy+\frac{3}{10}y^2$; $-\frac{3}{4}x^2+\frac{14}{15}xy-y^2$; $\frac{1}{2}x^2-xy+\frac{1}{5}y^2$.

28. $-\frac{3}{4}x^3+5ax^2-\frac{5}{8}a^2x$; $x^3-\frac{37}{8}ax^2+\frac{1}{2}a^2x$; $-\frac{1}{2}x^3+\frac{3}{4}a^2x$.

29. $\frac{3}{8}x^2-\frac{5}{3}xy-7y^2$; $\frac{2}{3}xy+\frac{18}{5}y^2$; $-\frac{5}{8}x^2+4y^2$.

30. $\frac{1}{2}a^3-2a^2b-\frac{3}{2}b^3$; $\frac{3}{2}a^2b-\frac{3}{4}ab^2+2b^3$; $-\frac{3}{2}a^3+ab^2+\frac{1}{2}b^3$.

# CHAPTER IV.

## SUBTRACTION.

26. THE simplest cases of subtraction have already come under the head of addition of *like* terms, of which some are negative. [Art. 19.]

Thus

$$\begin{aligned} 5a-3a &= 2a, \\ 3a-7a &= -4a, \\ -3a-6a &= -9a. \end{aligned}$$

Also, by the rule for removing brackets [Art. 22.],

$$\begin{aligned} 3a-(-8a) &= 3a+8a \\ &= 11a, \end{aligned}$$

and

$$\begin{aligned} -3a-(-8a) &= -3a+8a \\ &= 5a. \end{aligned}$$

27. The rule to be used when the expressions contain *unlike* terms is explained in the following example.

*Example.* Subtract $3a-2b-c$ from $4a-3b+5c$.

$$\begin{aligned} \text{The result of subtraction} &= 4a-3b+5c-(3a-2b-c) \\ &= 4a-3b+5c-3a+2b+c \\ &= 4a-3a-3b+2b+5c+c \\ &= a-b+6c. \end{aligned}$$

It is, however, more convenient to arrange the work as follows, the signs of all the terms in the lower line being changed.

$$\begin{array}{lrrr} & 4a & -3b & +5c \\ & -3a & +2b & +c \\ \hline \text{by } \textit{addition} & a & -b & +6c. \end{array}$$

**Rule.** *Change the sign of every term in the expression to be subtracted, and add to the other expression.*

**Note.** It is not necessary that in the expression to be subtracted the signs should be *actually* changed; the operation of changing signs ought to be performed mentally.

*Example* 1. From $5x^2+xy-3y^2$ take $2x^2+8xy-7y^2$.

$$\begin{array}{r}5x^2+\phantom{8}xy-3y^2\\ 2x^2+8xy-7y^2\\ \hline 3x^2-7xy+4y^2.\end{array}$$

*Example* 2. Subtract $2x^4-3x^2+7x-8$ from $x^4-2x^3-9x+4$.

$$\begin{array}{rrrrr}x^4 & -2x^3 & & -9x & +4\\ 2x^4 & & -3x^2 & +7x & -8\\ \hline -x^4 & -2x^3 & +3x^2 & -16x & +12.\end{array}$$

## EXAMPLES IV. a.

Subtract

1. $4a-3b+c$ from $2a-3b-c$.
2. $a-3b+5c$ from $4a-8b+c$.
3. $2x-8y+z$ from $15x+10y-18z$.
4. $15a-27b+8c$ from $10a+3b+4c$.
5. $-10x-14y+15z$ from $x-y-z$.
6. $-11ab+6cd$ from $-10bc+ab-4cd$.
7. $4a-3b+15c$ from $25a-16b-18c$.
8. $-16x-18y-15z$ from $-5x+8y+7z$.
9. $ab+cd-ac-bd$ from $ab+cd+ac+bd$.
10. $-ab+cd-ac+bd$ from $ab-cd+ac-bd$.

From

11. $3ab+5cd-4ac-6bd$ take $3ab+6cd-3ac-5bd$.
12. $yz-zx+xy$ take $-xy+yz-zx$.
13. $-2x^3-x^2-3x+2$ take $x^3-x+1$.
14. $-8x^2y+15xy^2+10xyz$ take $4x^2y-6xy^2-5xyz$.
15. $\frac{1}{2}a-b+\frac{1}{3}c$ take $\frac{1}{3}a+\frac{1}{2}b-\frac{1}{2}c$.
16. $\frac{3}{4}x+y-z$ take $\frac{1}{2}x-\frac{1}{2}y-\frac{1}{3}z$.
17. $-a-3b$ take $\frac{3}{2}a+\frac{1}{3}b-\frac{1}{2}c$.
18. $\frac{1}{2}x-\frac{3}{7}y+\frac{1}{10}z$ take $-\frac{1}{2}x+\frac{4}{7}y-\frac{1}{10}z$.
19. $-\frac{2}{3}x-\frac{3}{5}y-5z$ take $\frac{2}{3}x-\frac{3}{5}y-\frac{11}{3}z$.
20. $-\frac{1}{2}x+\frac{2}{3}y-\frac{1}{6}$ take $\frac{1}{3}x-\frac{3}{2}y-\frac{1}{6}$.

## EXAMPLES IV. b.

From

1. $3xy - 5yz + 8zx$ take $-4xy + 2yz - 10zx$.
2. $-8x^2y^2 + 15x^3y + 13xy^3$ take $4x^2y^2 + 7x^3y - 8xy^3$.
3. $-8 + 6ab + a^2b^2$ take $4 - 3ab - 5a^2b^2$.
4. $a^2bc + b^2ca + c^2ab$ take $3a^2bc - 5b^2ca - 4c^2ab$.
5. $-7a^2b + 8ab^2 + cd$ take $5a^2b - 7ab^2 + 6cd$.
6. $-8x^2y + 5xy^2 - x^2y^2$ take $8x^2y - 5xy^2 + x^2y^2$.
7. $10a^2b^2 + 15ab^2 + 8a^2b$ take $-10a^2b^2 + 15ab^2 - 8a^2b$.
8. $4x^2 - 3x + 2$ take $-5x^2 + 6x - 7$.
9. $x^3 + 11x^2 + 4$ take $8x^2 - 5x - 3$.
10. $-8a^2x^2 + 5x^2 + 15$ take $9a^2x^2 - 8x^2 - 5$.

Subtract

11. $x^3 - x^2 + x + 1$ from $x^3 + x^2 - x + 1$.
12. $3xy^2 - 3x^2y + x^3 - y^3$ from $x^3 + 3x^2y + 3xy^2 + y^3$.
13. $b^3 + c^3 - 2abc$ from $a^3 + b^3 - 3abc$.
14. $7xy^2 - y^3 - 3x^2y + 5x^3$ from $8x^3 + 7x^2y - 3xy^2 - y^3$.
15. $x^4 + 5 + x - 3x^3$ from $5x^4 - 8x^3 - 2x^2 + 7$.
16. $a^3 + b^3 + c^3 - 3abc$ from $7abc - 3a^3 + 5b^3 - c^3$.
17. $1 - x + x^5 - x^4 - x^3$ from $x^4 - 1 + x - x^2$.
18. $7a^4 - 8a^2 + 3a^5 + a$ from $a^2 - 5a^3 - 7 + 7a^5$.
19. $10a^2b + 8ab^2 - 8a^3b^3 - b^4$ from $5a^2b - 6ab^2 - 7a^3b^3$.
20. $a^3 - b^3 + 8ab^2 - 7a^2b$ from $-8ab^2 + 15a^2b + b^3$.

From

21. $\frac{1}{2}x^2 - \frac{1}{3}xy - \frac{3}{2}y^2$ take $-\frac{3}{2}x^2 + xy - y^2$.
22. $\frac{2}{3}a^2 - \frac{5}{2}a - 1$ take $-\frac{2}{3}a^2 + a - \frac{1}{2}$.
23. $\frac{1}{3}x^2 - \frac{1}{2}x + \frac{1}{6}$ take $\frac{1}{3}x - 1 + \frac{1}{2}x^2$.
24. $\frac{3}{8}x^2 - \frac{2}{3}ax$ take $\frac{1}{3} - \frac{1}{4}x^2 - \frac{5}{6}ax$.
25. $\frac{3}{4}x^3 - \frac{1}{3}xy^2 - y^2$ take $\frac{1}{2}x^2y - \frac{5}{6}y^2 - \frac{1}{3}xy^2$.
26. $\frac{1}{8}a^3 - 2ax^2 - \frac{1}{3}a^2x$ take $\frac{1}{3}a^2x + \frac{1}{4}a^3 - \frac{3}{2}ax^2$.

28. We shall close this chapter with an exercise containing miscellaneous examples of Addition and Subtraction.

## EXAMPLES IV. c.

**1.** To the sum of $2a-3b-2c$ and $2b-a+7c$ add the sum of $a-4c+7b$ and $c-6b$.

**2.** From $5x^3+3x-1$ take the sum of $2x-5+7x^2$ and $3x^2+4-2x^3+x$.

**3.** Subtract $3a-7a^3+5a^2$ from the sum of $2+8a^2-a^3$ and $2a^3-3a^2+a-2$.

**4.** Subtract $5x^2+3x-1$ from $2x^3$, and add the result to $3x^2+3x-1$.

**5.** Add the sum of $2y-3y^2$ and $1-5y^3$ to the remainder left when $1-2y^2+y$ is subtracted from $5y^3$.

**6.** Take $x^2-y^2$ from $3xy-4y^2$, and add the remainder to the sum of $4xy-x^2-3y^2$ and $2x^2+6y^2$.

**7.** Find the sum of $5a-7b+c$ and $3b-9a$, and subtract the result from $c-4b$.

**8.** Add together $3x^2-7x+5$ and $2x^3+5x-3$, and diminish the result by $3x^2+2$.

**9.** What expression must be added to $5x^2-7x+2$ to produce $7x^2-1$?

**10.** What expression must be added to $4x^3-3x^2+2$ to produce $4x^3+7x-6$?

**11.** What expression must be subtracted from $3a-5b+c$ so as to leave $2a-4b+c$?

**12.** What expression must be subtracted from $9x^2+11x-5$ so as to leave $6x^2-17x+3$?

**13.** From what expression must $11a^2-5ab-7bc$ be subtracted so as to give for remainder $5a^2+7ab+7bc$?

**14.** From what expression must $3ab+5bc-6ca$ be subtracted so as to leave a remainder $6ca-5bc$?

**15.** To what expression must $7x^3-6x^2-5x$ be added so as to make $9x^3-6x-7x^2$?

**16.** To what expression must $5ab-11bc-7ca$ be added so as to produce zero?

**17.** If $3x^2-7x+2$ be subtracted from zero, what will be the result?

**18.** Subtract $3x^3-7x+1$ from $2x^2-5x-3$, then subtract the difference from zero, and add this last result to $2x^2-2x^3-4$.

**19.** Subtract $3x^2-5x+1$ from unity, and add $5x^2-6x$ to the result.

# CHAPTER V.

## MULTIPLICATION.

29. WHEN there is no sign between symbols or expressions, it is understood that the symbols or expressions are to be *multiplied* together.

Thus

$$ab = a \times b.$$

$$3ab = 3 \times a \times b.$$

$$a(x-y) = a \times (x-y).$$

$(x+y)(a+b)$ is the *product* of $x+y$ and $a+b$.

In Algebra, as in Arithmetic, the product is the same in whatever order the factors are written. [Art. 13.]

*Example.*

$$\begin{aligned} 2a \times 3b &= 2 \times a \times 3 \times b \\ &= 2 \times 3 \times a \times b \\ &= 6ab. \end{aligned}$$

30. Since, by definition, $a^3 = aaa$,

and

$$a^5 = aaaaa;$$

$$\begin{aligned} \therefore \quad a^3 \times a^5 &= aaaaaaaa \\ &= a^8. \end{aligned}$$

Again,

$$5a^2 = 5aa,$$

$$7a^3 = 7aaa;$$

$$\begin{aligned} \therefore \quad 5a^2 \times 7a^3 &= 5 \times 7 \times aaaaa \\ &= 35a^5. \end{aligned}$$

31. When the expressions to be multiplied together contain powers of different letters, a similar method is used.

*Example.*

$$\begin{aligned} 5a^3b^2 \times 8a^2bx^3 &= 5aaabb \times 8aabxxx \\ &= 40a^5b^3x^3. \end{aligned}$$

**Note.** The beginner must be careful to observe that in this process of multiplication the indices of one letter cannot combine in any way with those of another. Thus the expression $40a^5b^3x^3$ admits of no further simplification.

32. The general rule may now be given:

**Rule.** *To multiply two simple expressions together, multiply their coefficients, and add the indices of the like letters.*

The rule may be extended to cases where more than two expressions are to be multiplied together.

*Example* 1.
$$\begin{aligned} x^2 \times x^3 \times x^8 &= x^{2+3} \times x^8 \\ &= x^{2+3+8} \\ &= x^{13}. \end{aligned}$$

*Example* 2. $5x^2y^3 \times 8y^2z^5 \times 3xz^4 = 120x^3y^5z^9.$

**Note.** The product of three or more expressions is called **the continued product.**

33. The symbol $4m$ means *four times m*, that is
$$m+m+m+m.$$

So $\quad 10m = m+m+m+ \ldots\ldots$ taking ten terms.

Similarly $am$ means that $m$ is to be taken $a$ times; that is
$$am = m+m+m+ \ldots\ldots$$
the number of terms being $a$.

Again, $(a+b)\,m = m+m+m+ \ldots\ldots$ taken $a+b$ times

$= (m+m+m+ \ldots\ldots$ taken $a$ times),

together with $(m+m+m+ \ldots\ldots$ taken $b$ times)

$= am+bm \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$

Also $(a-b)\,m = m+m+m+ \ldots\ldots$ taken $a-b$ times

$= (m+m+m+ \ldots\ldots$ taken $a$ times),

diminished by $(m+m+m+ \ldots\ldots$ taken $b$ times)

$= am-bm \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$

Similarly $\qquad (a-b+c)\,m = am-bm+cm.$

34. **Rule.** *To multiply a compound expression by a single factor, multiply each term of the expression separately by that factor.*

*Examples.*

(1) $3\,(2a+3b-4c) = 6a+9b-12c.$

(2) $(4x^2-7y-8z^3) \times 3xy^2 = 12x^3y^2-21xy^3-24xy^2z^3.$

(3) $(\frac{2}{3}a-\frac{1}{6}b-c) \times \frac{3}{8}ab^2 = \frac{1}{4}a^2b^2-\frac{1}{16}ab^3-\frac{3}{8}ab^2c.$

## EXAMPLES V. a.

Multiply together

**1.** $5x^2$ and $7x^5$. **2.** $4a^3$ and $5a^8$.
**3.** $7ab$ and $8a^3b^2$. **4.** $6xy^2$ and $5x^3$.
**5.** $8a^3b$ and $b^5$. **6.** $2abc$ and $3ac^3$.
**7.** $2a^3b^3$ and $2a^3b^3$. **8.** $5a^2b$ and $2a$.
**9.** $4a^2b^3$ and $7a^5$. **10.** $5a^4b^3$ and $x^2y^2$.
**11.** $x^3y^3$ and $6a^2x^4$. **12.** $abc$ and $xyz$.
**13.** $3a^4b^7x^3$ and $5a^3bx$. **14.** $4a^3bx$ and $7b^2x^4$.
**15.** $5a^2x$ and $8cx$. **16.** $5x^3y^3$ and $6a^3x^3$.
**17.** $2x^2y$ and $x^5y^7$. **18.** $3a^3x^4y^7$ and $a^2x^5y^9$.
**19.** $ab+bc$ and $a^3b$. **20.** $5ab-7bx$ and $4a^2bx^3$.
**21.** $5x+3y$ and $2x^2$. **22.** $a^2+b^2-c^2$ and $a^3b$.
**23.** $bc+ca-ab$ and $abc$. **24.** $5a^2+3b^2-2c^2$ and $4a^2bc^3$.
**25.** $5x^2y+xy^2-7x^2y^2$ and $3x^3$.
**26.** $6x^3-5x^2y+7xy^2$ and $8x^2y^3$.
**27.** $6a^3bc-7ab^2c^2$ and $a^2b^2$. **28.** $8a^2b^3-\frac{2}{3}b^2c^3$ and $\frac{1}{4}ab^4$.
**29.** $a^4b^3x-5b^6$ and $\frac{1}{5}a^3x^5$. **30.** $6a^3-5a^2b^3x^4$ and $\frac{3}{2}a^4b^5$.

35. If in Art. 33 we write $c+d$ for $m$ in (1), we have

$$\begin{aligned}(a+b)(c+d)&=a(c+d)+b(c+d)\\&=(c+d)a+(c+d)b \qquad \text{[Art. 13.]},\\&=ac+ad+bc+bd \quad \ldots\ldots\ldots\ldots\ldots(3).\end{aligned}$$

Again, from (2)

$$\begin{aligned}(a-b)(c+d)&=a(c+d)-b(c+d)\\&=(c+d)a-(c+d)b\\&=ac+ad-(bc+bd)\\&=ac+ad-bc-bd \quad \ldots\ldots\ldots\ldots\ldots(4).\end{aligned}$$

Similarly, by writing $c-d$ for $m$ in (1),

$$\begin{aligned}(a+b)(c-d)&=a(c-d)+b(c-d)\\&=(c-d)a+(c-d)b\\&=ac-ad+bc-bd \quad \ldots\ldots\ldots\ldots\ldots(5).\end{aligned}$$

Also, from (2)

$$\begin{aligned}(a-b)(c-d) &= a(c-d)-b(c-d)\\ &= (c-d)a-(c-d)b\\ &= ac-ad-(bc-bd)\\ &= ac-ad-bc+bd \quad \ldots\ldots\ldots\ldots\ldots(6).\end{aligned}$$

These results are very important.

If we consider each term on the right-hand side of (6), and the way in which it arises, we find that

$$\begin{aligned}+a \times +c &= +ac.\\ -b \times -d &= +bd.\\ -b \times +c &= -bc.\\ +a \times -d &= -ad.\end{aligned}$$

These results enable us to state what is known as the **Rule of Signs** in multiplication.

RULE OF SIGNS. *The product of two terms with like signs is positive; the product of two terms with unlike signs is negative.*

36. The rule of signs, and especially the use of the negative multiplier, will probably present some difficulty to the beginner. Perhaps the following numerical instances may be useful in illustrating the interpretation that may be given to multiplication by a negative quantity.

$$\begin{aligned}-3 \times +4 &= -3 \times 4\\ &= -3 \text{ taken four times}\\ &= -3-3-3-3\\ &= -12.\end{aligned}$$

$+3 \times -4$ would thus seem to mean that $+3$ is to be taken $-4$ times; this is at present without meaning, but we can easily give to it an interpretation which will be intelligible, and in harmony with what precedes. We have illustrated the difference between $+4$ and $-4$, by supposing that $+4$ represents a line of 4 units measured in one direction, and $-4$ a line of length 4 units measured in the *opposite* direction; so that, with reference to each other, each of the signs $+$ and $-$ may be supposed to indicate a *reversal of direction.* Hence $3 \times -4$ may be taken as indicating that 3 is to be taken 4 times, and further that the direction of the line which represents the product is to be reversed.

Now 3 taken 4 times gives $+12$; and from this by changing the sign, which corresponds to a reversal of direction, we get $-12$.

Thus $$3\times -4=-12.$$

Similarly $-3\times -4$ indicates that $-3$ is to be taken 4 times, and the sign changed; the first operation gives $-12$, and the second $+12$.

Thus $$-3\times -4=+12.$$

37. We may here remark that we can give what meaning we like to any new symbol or operation, provided that we *always* employ such meaning, and that the meaning is not inconsistent with the fundamental principles of our subject. Multiplication by a negative quantity is a case in point; and the meaning we have attached to the operation is to proceed just as if the multiplier were positive, and then change the sign of the product.

38. To familiarize the beginner with the principles we have just explained we add a few examples in substitutions where some of the symbols denote negative quantities.

*Example* 1. If $a=-4$, $a^3=(-4)^3=-4\times -4\times -4=-64$.

*Example* 2. If $a=-1$, $b=3$, $c=-2$, find the value of $-3a^4bc^3$.

$$\begin{aligned}-3a^4bc^3&=-3\times(-1)^4\times 3\times(-2)^3\\&=-3\times 1\times 3\times(-8)\\&=72.\end{aligned}$$

## EXAMPLES V. b.

If $a=-2$, $b=3$, $c=-1$, $x=-5$, $y=4$, find the value of

1. $3a^2b$.
2. $8abc^2$.
3. $-5c^3$.
4. $6a^2c^2$.
5. $4c^3y$.
6. $3a^2c$.
7. $-b^2c^2$.
8. $3a^3c^2$.
9. $-7a^3bc$.
10. $-2a^4bx$.
11. $-4a^2c^4$.
12. $3c^3x^3$.
13. $5a^2x^2$.
14. $-7c^4xy$.
15. $-8ax^3$.
16. $4c^5x^3$.
17. $-5a^2b^2c^2$.
18. $-7a^3c^3$.
19. $8c^4x^3$.
20. $7a^5c^4$.

If $a=-4$, $b=-3$, $c=-1$, $f=0$, $x=4$, $y=1$, find the value of

21. $3a^2+bx-4cy$.
22. $2ab^2-3bc^2+2fx$.
23. $fa^2-2b^3-cx^3$.
24. $3a^2y^3-5b^2x-2c^3$.
25. $2a^3-3b^3+7cy^4$.
26. $3b^2y^4-4b^2f-6c^4x$.
27. $2\sqrt{(ac)}-3\sqrt{(xy)}+\sqrt{(b^2c^4)}$.
28. $3\sqrt{(acx)}-2\sqrt{(b^2y)}-6\sqrt{(c^2y)}$.
29. $7\sqrt{(a^2x)}-3\sqrt{(b^4c^2)}+5\sqrt{(f^2x)}$.
30. $3c\sqrt{(3bc)}-5\sqrt{(4c^2y^3)}-2cy\sqrt{(3bc^5)}$.

39. The following examples further illustrate the rule of signs and the law of indices:

*Example* 1. $4a^2b \times -3a^4b^2 = -12a^6b^3$.

*Example* 2. $-5ab^3 \times -ab^3 = 5a^2b^6$.

*Example* 3. Find the continued product of $3a^2b$, $-2a^3b^2$, $-ab^4$.

$$3a^2b \times -2a^3b^2 = -6a^5b^3,$$

and

$$-6a^5b^3 \times -ab^4 = 6a^6b^7.$$

*Example* 4. $(6a^3 - 5a^2b - 4ab^2) \times -3ab^3 = -18a^4b^3 + 15a^3b^4 + 12a^2b^5$.

## EXAMPLES V. c.

Multiply together

**1.** $ax$ and $-3ax$. **2.** $-2abx$ and $-7abx$.
**3.** $a^2b$ and $-ab^2$. **4.** $6x^2y$ and $-10xy$.
**5.** $-abcd$ and $-3a^2b^3c^4d^5$. **6.** $xyz$ and $-5x^2y^3z$.
**7.** $3xy + 4yz$ and $-12xyz$. **8.** $ab - bc$ and $a^2bc^3$.
**9.** $-x - y - z$ and $-3x$. **10.** $a^2 - b^2 + c^2$ and $abc$.
**11.** $-ab + bc - ca$ and $-abc$. **12.** $-2a^2b - 4ab^2$ and $-7a^2b^2$.
**13.** $5x^2y - 6xy^2 + 8x^2y^2$ and $3xy$.
**14.** $-7x^3y - 5xy^3$ and $-8x^3y^3$.
**15.** $-5xy^2z + 3xyz^2 - 8x^2yz$ and $xyz$.
**16.** $4x^2y^2z^2 - 8xyz$ and $-12x^3yz^3$.
**17.** $-13xy^2 - 15x^2y$ and $-7x^3y^3$.
**18.** $8xyz - 10x^3yz^3$ and $-xyz$.
**19.** $abc - a^2bc - ab^2c$ and $-abc$.
**20.** $-a^2bc + b^2ca - c^2ab$ and $-ab$.

Find the product of

**21.** $2a - 3b + 4c$ and $-\frac{3}{2}a$. **22.** $3x - 2y - 4$ and $-\frac{5}{6}x$.
**23.** $\frac{2}{3}a - \frac{1}{6}b - c$ and $\frac{3}{8}ax$. **24.** $\frac{6}{7}a^2x^2 - \frac{3}{2}ax^3$ and $-\frac{7}{3}a^3x$.
**25.** $-\frac{5}{3}a^2x^2$ and $-\frac{3}{2}a^2 + ax - \frac{3}{5}x^2$.
**26.** $-\frac{7}{2}xy$ and $-3x^2 + \frac{2}{7}xy$. **27.** $-\frac{3}{2}x^3y^2$ and $-\frac{1}{3}x^2 + 2y^2$.
**28.** $-\frac{4}{7}x^5y^3$ and $\frac{7}{4}x^3 - \frac{4}{7}y^3$.

40. The complete rule for multiplying together two compound expressions may now be given.

**Rule.** *Multiply each term of the first expression by each term of the second. When the terms multiplied together have like signs, prefix to the product the sign +, when unlike prefix −.*

*Example* 1. Multiply $x+8$ by $x+7$.

$$\begin{array}{ll}
\text{The product} & =(x+8)\times(x+7) \\
 & =x^2+8x+7x+56 \\
 & =x^2+15x+56.
\end{array}$$

The operation is more conveniently arranged as follows:

$$\begin{array}{llll}
 & x & +\ 8 & \\
 & x & +\ 7 & \\
\cline{2-3}
 & x^2 & +\ 8x & \\
 & & +\ 7x & +56 \\
\cline{2-4}
\text{by addition} & x^2 & +15x & +56.
\end{array}$$

**Note.** We begin on the left and work to the right, placing the second result one place to the right, so that like terms may stand in the same vertical column.

*Example* 2. Multiply $2x-3y$ by $4x-7y$.

$$\begin{array}{llll}
 & 2x & -3y & \\
 & 4x & -7y & \\
\cline{2-3}
 & 8x^2 & -12xy & \\
 & & -14xy & +21y^2 \\
\cline{2-4}
\text{by addition} & 8x^2 & -26xy & +21y^2.
\end{array}$$

## EXAMPLES V. d.

Find the product of

1. $x+5$ and $x+10$.
2. $x+5$ and $x-5$.
3. $x-7$ and $x-10$.
4. $x-7$ and $x+10$.
5. $x+7$ and $x-10$.
6. $x+7$ and $x+10$.
7. $x+6$ and $x-6$.
8. $x+8$ and $x-4$.
9. $x-12$ and $x-1$.
10. $x+12$ and $x-1$.
11. $x-15$ and $x+15$.
12. $x-15$ and $-x+3$.
13. $-x-2$ and $-x-3$.
14. $-x+7$ and $x-7$.
15. $-x+5$ and $-x-5$.
16. $x-13$ and $x+14$.
17. $x-17$ and $x+18$.
18. $x+19$ and $x-20$.
19. $-x-16$ and $-x+16$.
20. $-x+21$ and $x-21$.
21. $2x-3$ and $x+8$.
22. $2x+3$ and $x-8$.

**23.** $x-5$ and $2x-1$.
**24.** $2x-5$ and $x-1$.
**25.** $3x-5$ and $2x+7$.
**26.** $3x+5$ and $2x-7$.
**27.** $5x-6$ and $2x+3$.
**28.** $5x+6$ and $2x-3$.
**29.** $3x-5y$ and $3x+5y$.
**30.** $3x-5y$ and $3x-5y$.
**31.** $a-2b$ and $a+3b$.
**32.** $a-7b$ and $a+8b$.
**33.** $3a-6b$ and $a-8b$.
**34.** $a-9b$ and $a+5b$.
**35.** $x+a$ and $x-b$.
**36.** $x-a$ and $x+b$.
**37.** $x-2a$ and $x+3b$.
**38.** $ax-by$ and $ax+by$.
**39.** $xy-ab$ and $xy+ab$.
**40.** $2pq-3r$ and $2pq+3r$.

41. When the expressions to be multiplied together contain more than two terms, a similar method may be employed.

*Example.* Find the product of $2a^2-3ab+4b^2$ and $-5a^2+3ab+4b^2$.

$$\begin{array}{rrrrr}
2a^2 & -\ 3ab & +4b^2 & & \\
-5a^2 & +\ 3ab & +4b^2 & & \\
\hline
-10a^4 & +15a^3b & -20a^2b^2 & & \\
 & +\ 6a^3b & -\ 9a^2b^2 & +12ab^3 & \\
 & & 8a^2b^2 & -12ab^3 & +16b^4 \\
\hline
-10a^4 & +21a^3b & -21a^2b^2 & & +16b^4.
\end{array}$$

42. If multiplier and multiplicand are not arranged according to powers ascending or descending of some common letter, a re-arrangement will be found convenient.

*Example.* Multiply $2xz-z^2+2x^2-3yz+xy$ by $x-y+2z$.

$$\begin{array}{lllllllll}
2x^2 & +\ xy & +2xz & -3yz & -z^2 & & & & \\
x & -\ y & +2z & & & & & & \\
\hline
2x^3 & +\ x^2y & +2x^2z & -3xyz & -xz^2 & & & & \\
 & -2x^2y & & -2xyz & & -xy^2 & +3y^2z & +yz^2 & \\
 & & 4x^2z & +2xyz & +4xz^2 & & & -6yz^2 & -2z^3 \\
\hline
2x^3 & -\ x^2y & +6x^2z & -3xyz & +3xz^2 & -xy^2 & +3y^2z & -5yz^2 & -2z^3.
\end{array}$$

43. When the coefficients are fractional we use the ordinary process of Multiplication, combining the fractional coefficients by the rules of Arithmetic.

*Example.* Multiply $\frac{1}{3}a^2-\frac{1}{2}ab+\frac{2}{3}b^2$ by $\frac{1}{2}a+\frac{1}{3}b$.

$$\begin{array}{lllll}
\frac{1}{3}a^2 & -\ \frac{1}{2}ab & +\frac{2}{3}b^2 & & \\
\frac{1}{2}a & +\ \frac{1}{3}b & & & \\
\hline
\frac{1}{6}a^3 & -\ \frac{1}{4}a^2b & +\frac{1}{3}ab^2 & & \\
 & +\ \frac{1}{9}a^2b & -\frac{1}{6}ab^2 & +\frac{2}{9}b^3 & \\
\hline
\frac{1}{6}a^3 & -\frac{5}{36}a^2b & +\frac{1}{6}ab^2 & +\frac{2}{9}b^3. &
\end{array}$$

## EXAMPLES V. e.

Multiply together

1. $a+b+c$ and $a+b-c$. 2. $a-2b+c$ and $a+2b-c$.
3. $a^2-ab+b^2$ and $a^2+ab+b^2$.
4. $x^2+3y^2$ and $x+4y$. 5. $x^3-2x^2+8$ and $x+2$.
6. $x^4-x^2y^2+y^4$ and $x^2+y^2$. 7. $x^2+xy+y^2$ and $x-y$.
8. $a^2-2ax+4x^2$ and $a^2+2ax+4x^2$.
9. $16a^2+12ab+9b^2$ and $4a-3b$.
10. $a^2x-ax^2+x^3-a^3$ and $x+a$.
11. $x^2+x-2$ and $x^2+x-6$.
12. $2x^3-3x^2+2x$ and $2x^2+3x+2$.
13. $-a^5+a^4b-a^3b^2$ and $-a-b$.
14. $x^3-7x+5$ and $x^2-2x+3$.
15. $a^3+2a^2b+2ab^2$ and $a^2-2ab+2b^2$.
16. $4x^2+6xy+9y^2$ and $2x-3y$.
17. $x^2-3xy-y^2$ and $-x^2+xy+y^2$.
18. $b^3-a^2b^2+a^3$ and $a^3+a^2b^2+b^3$.
19. $x^2-2xy+y^2$ and $x^2+2xy+y^2$.
20. $ab+cd+ac+bd$ and $ab+cd-ac-bd$.
21. $-3a^2b^2+4ab^3+15a^3b$ and $5a^2b^2+ab^3-3b^4$.
22. $27x^3-36ax^2+48a^2x-64a^3$ and $3x+4a$.
23. $a^2-5ab-b^2$ and $a^2+5ab+b^2$.
24. $x^2-xy+x+y^2+y+1$ and $x+y-1$.
25. $a^2+b^2+c^2-bc-ca-ab$ and $a+b+c$.
26. $-x^3y+y^4+x^2y^2+x^4-xy^3$ and $x+y$.
27. $x^{12}-x^9y^2+x^6y^4-x^3y^6+y^8$ and $x^3+y^2$.
28. $3a^2+2a+2a^3+1+a^4$ and $a^2-2a+1$.
29. $-ax^2+3axy^2-9ay^4$ and $-ax-3ay^2$.
30. $-2x^3y+y^4+3x^2y^2+x^4-2xy^3$ and $x^2+2xy+y^2$.
31. $\frac{1}{2}a^2+\frac{1}{3}a+\frac{1}{4}$ and $\frac{1}{2}a-\frac{1}{3}$. 32. $\frac{1}{2}x^2-2x+\frac{3}{2}$ and $\frac{1}{2}x+\frac{1}{3}$.
33. $\frac{2}{3}x^2+xy+\frac{3}{2}y^2$ and $\frac{1}{3}x-\frac{1}{2}y$.
34. $\frac{3}{2}x^2-ax-\frac{2}{3}a^2$ and $\frac{3}{4}x^2-\frac{1}{2}ax+\frac{1}{3}a^2$.
35. $\frac{1}{2}x^2-\frac{2}{3}x-\frac{3}{4}$ and $\frac{1}{2}x^2+\frac{2}{3}x-\frac{3}{4}$.
36. $\frac{2}{3}ax+\frac{2}{3}x^2+\frac{1}{3}a^2$ and $\frac{3}{4}a^2+\frac{3}{2}x^2-\frac{3}{2}ax$.

44. Although the result of multiplying together two binomial factors, such as $x+8$ and $x-7$, can always be obtained by the methods already explained, it is of the utmost importance that the student should soon learn to write down the product rapidly *by inspection.*

This is done by observing in what way the coefficients of the terms in the product arise, and noticing that they result from the combination of the numerical coefficients in the two binomials which are multiplied together; thus

$$\begin{aligned}(x+8)(x+7)&=x^2+8x+7x+56\\&=x^2+15x+56.\\(x-8)(x-7)&=x^2-8x-7x+56\\&=x^2-15x+56.\\(x+8)(x-7)&=x^2+8x-7x-56\\&=x^2+x-56.\\(x-8)(x+7)&=x^2-8x+7x-56\\&=x^2-x-56.\end{aligned}$$

In each of these results we notice that:

1. The product consists of three terms.

2. The first term is the product of the first terms of the two binomial expressions.

3. The third term is the product of the second terms of the two binomial expressions.

4. The middle term has for its coefficient the sum of the numerical quantities (taken with their proper signs) in the second terms of the two binomial expressions.

The intermediate step in the work may be omitted, and the products written down at once, as in the following examples:

$$\begin{aligned}(x+2)(x+3)&=x^2+5x+6.\\(x-3)(x+4)&=x^2+x-12.\\(x+6)(x-9)&=x^2-3x-54.\\(x-4y)(x-10y)&=x^2-14xy+40y^2.\\(x-6y)(x+4y)&=x^2-2xy-24y^2.\end{aligned}$$

By an easy extension of these principles we may write down the product of *any* two binomials.

Thus
$$\begin{aligned}(2x+3y)(x-y)&=2x^2+3xy-2xy-3y^2\\&=2x^2+xy-3y^2.\\(3x-4y)(2x+y)&=6x^2-8xy+3xy-4y^2\\&=6x^2-5xy-4y^2.\end{aligned}$$

45. These examples will be sufficient, but there are two cases to which it is desirable to direct the student's attention before leaving this part of the subject: their full importance will be developed in later chapters.

CASE I.
$$\begin{aligned}(x+4)(x-4)&=x^2+4x-4x-16\\&=x^2-16.\\(2x+5y)(2x-5y)&=4x^2+10xy-10xy-25y^2\\&=4x^2-25y^2.\end{aligned}$$

CASE II.
$$\begin{aligned}(x+3)^2&=(x+3)(x+3)\\&=x^2+6x+9.\\(3x-4y)^2&=(3x-4y)(3x-4y)\\&=9x^2-12xy-12xy+16y^2\\&=9x^2-24xy+16y^2.\end{aligned}$$

## EXAMPLES V. f.

*Write down* the values of the following products:

**1.** $(x+8)(x-5)$.
**2.** $(x+6)(x-1)$.
**3.** $(x-3)(x+10)$.
**4.** $(x-1)(x+5)$.
**5.** $(x+7)(x-9)$.
**6.** $(x-10)(x-8)$.
**7.** $(x-4)(x+11)$.
**8.** $(x-2)(x+4)$.
**9.** $(x+2)(x-2)$.
**10.** $(a-1)(a+1)$.
**11.** $(a+9)(a-5)$.
**12.** $(a-3)(a+12)$.
**13.** $(a-8)(a+4)$.
**14.** $(a-8)(a+8)$.
**15.** $(a-6)(a+13)$.
**16.** $(a+3)(a+3)$.
**17.** $(a-11)(a+11)$.
**18.** $(a-8)(a-8)$.
**19.** $(x-3a)(x+2a)$.
**20.** $(x+6a)(x-5a)$.
**21.** $(x+3a)(x-3a)$.
**22.** $(x+4y)(x-2y)$.
**23.** $(x+7y)(x-7y)$.
**24.** $(x-3y)(x-3y)$.
**25.** $(a+3b)(a+3b)$.
**26.** $(a-5b)(a+10b)$.
**27.** $(a-9b)(a-8b)$.
**28.** $(2x-5)(x+2)$.
**29.** $(2x-5)(x-2)$.
**30.** $(2x+3)(x-3)$.
**31.** $(3x-1)(x+1)$.
**32.** $(2x+5)(2x-1)$.
**33.** $(3x+7)(2x-3)$.
**34.** $(4x-3)(2x+3)$.
**35.** $(3x+8)(3x-8)$.
**36.** $(2x-5)(2x-5)$.
**37.** $(3x-2y)(3x+y)$.
**38.** $(3x+2y)(3x+2y)$.
**39.** $(2x+7y)(2x-5y)$.
**40.** $(5x+3a)(5x-3a)$.
**41.** $(2x-5a)(x+5a)$.
**42.** $(2x+a)(2x+a)$.

# CHAPTER VI.

## DIVISION.

46. THE object of division is to find out the quantity, called the **quotient**, by which the **divisor** must be multiplied so as to produce the **dividend**.

Division is thus the inverse of multiplication.

The above statement may be briefly written

$$\text{quotient} \times \text{divisor} = \text{dividend},$$

or

$$\text{dividend} \div \text{divisor} = \text{quotient}.$$

It is sometimes better to express this last result as a fraction; thus

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient}.$$

*Example* 1. Since the product of 4 and $x$ is $4x$, it follows that when $4x$ is divided by $x$ the quotient is 4,

or otherwise, $$4x \div x = 4.$$

*Example* 2. $27a^5 \div 9a^3 = \frac{27a^5}{9a^3} = \frac{27aaaaa}{9aaa} = 3aa$, by removing from the divisor and dividend the factors common to both, just as in Arithmetic.

Therefore $$27a^5 \div 9a^3 = 3a^2.$$

*Example* 3. $$35a^3b^2c^3 \div 7ab^2c^2 = \frac{35aaabbccc}{7abbcc}$$
$$= 5a^2c.$$

47. RULE. *To divide one simple expression by another, divide the coefficient of the dividend by that of the divisor, and subtract the index of any letter in the divisor from the index of that letter in the dividend.*

*Example* 4. $84a^5x^3 \div 12a^4x = 7a^{5-4}x^{3-1}$
$= 7ax^2$.

*Example* 5. $77a^2x^3y^4 \div 7ax^2y = 11axy^3$.

NOTE. If we apply the rule to divide any power of a letter by the same power of the letter we are led to a curious conclusion.

Thus, by the rule $a^3 \div a^3 = a^{3-3} = a^0$;

but also $a^3 \div a^3 = \dfrac{a^3}{a^3} = 1$,

$\therefore\ a^0 = 1$.

This result will appear somewhat strange to the beginner, but its full significance will be explained in chap. xxx.

48. It is easy to prove that *the rule of signs holds for division.*

Thus
$$ab \div a = \frac{ab}{a} = \frac{a \times b}{a} = b.$$
$$-ab \div a = \frac{-ab}{a} = \frac{a \times -b}{a} = -b.$$
$$ab \div -a = \frac{ab}{-a} = \frac{-a \times -b}{-a} = -b.$$
$$-ab \div -a = \frac{-ab}{-a} = \frac{-a \times b}{-a} = b.$$

Hence in division as well as multiplication
*like signs produce* $+$,
*unlike signs produce* $-$.

*Examples.* (1) $6ab \div 2a = 3b$.
(2) $-15xy \div 3x = -5y$.
(3) $-21a^2b^3 \div -7a^2b^2 = 3b$.
(4) $45a^6b^2x^4 \div -9a^3bx^2 = -5a^3bx^2$.
(5) $-\frac{4}{5}a^4b^5c^3 \div -\frac{3}{5}a^3b^3c^3 = \frac{1}{3}ab^2$.

49. RULE. *To divide a compound expression by a single factor, divide each term separately by that factor.*

This follows at once from Art. 34.

*Examples.* (1) $(9x - 12y + 3z) \div -3 = -3x + 4y - z$.
(2) $(36a^3b^2 - 24a^2b^5 - 20a^4b^2) \div 4a^2b = 9ab - 6b^4 - 5a^2b$.
(3) $(2x^2 - 5xy + \frac{3}{2}x^2y^3) \div -\frac{1}{2}x = -4x + 10y - 3xy^3$.

## EXAMPLES VI. a.

Divide

**1.** $3x^3$ by $x^2$. **2.** $27x^4$ by $-9x^3$.

**3.** $-35x^6$ by $7x^3$. **4.** $abx^2$ by $-ax$.

**5.** $x^3y^3$ by $x^2y$. **6.** $a^4x^3$ by $-a^2x^3$.

**7.** $4a^2b^2c^3$ by $ab^2c^2$. **8.** $12a^6b^6c^6$ by $-3a^4b^2c$.

**9.** $-a^5c^9$ by $-ac^3$. **10.** $15x^5y^7z^4$ by $5x^2y^2z^2$.

**11.** $-16x^3y^2$ by $-4xy^2$. **12.** $-48a^9$ by $-8a^3$.

**13.** $35a^{11}$ by $7a^7$. **14.** $63a^7b^8c^3$ by $9a^5b^5c^3$.

**15.** $7a^2bc$ by $-7a^2bc$. **16.** $28a^4b^3$ by $-4a^3b$.

**17.** $16b^2yx^2$ by $-2xy$. **18.** $-50y^3x^3$ by $-5x^3y$.

**19.** $x^2-2xy$ by $x$. **20.** $x^3-3x^2+x$ by $x$.

**21.** $x^6-7x^5+4x^4$ by $x^2$. **22.** $10x^7-8x^6+3x^4$ by $x^3$.

**23.** $15x^5-25x^4$ by $-5x^3$. **24.** $27x^6-36x^5$ by $9x^5$.

**25.** $-24x^6-32x^4$ by $-8x^3$. **26.** $34x^3y^2-51x^2y^3$ by $17xy$.

**27.** $a^2-ab-ac$ by $-a$. **28.** $a^3-a^2b-a^2b^2$ by $a^2$.

**29.** $3x^3-9x^2y-12xy^2$ by $-3x$.

**30.** $4x^4y^4-8x^3y^2+6xy^3$ by $-2xy$.

**31.** $-3a^2+\frac{9}{2}ab-6ac$ by $-\frac{3}{2}a$. **32.** $\frac{1}{2}x^5y^2-3x^3y^4$ by $-\frac{3}{2}x^3y^2$.

**33.** $-\frac{5}{2}x^2+\frac{5}{3}xy+\frac{10}{3}x$ by $-\frac{5}{6}x$.

**34.** $-2a^5x^3+\frac{7}{2}a^4x^4$ by $\frac{7}{3}a^3x$.

**35.** $\frac{1}{4}a^2x-\frac{1}{16}abx-\frac{3}{8}acx$ by $\frac{3}{8}ax$.

50. To divide one compound expression by another.

RULE. 1. *Arrange divisor and dividend in ascending or descending powers of some common letter.*

2. *Divide the term on the left of the dividend by the term on the left of the divisor, and put the result in the quotient.*

3. *Multiply the* WHOLE *divisor by this quotient, and put the product under the dividend.*

4. *Subtract and bring down from the dividend as many terms as may be necessary.*

*Repeat these operations till all the terms from the dividend are brought down.*

*Example* 1. Divide $x^2+11x+30$ by $x+6$.

Arrange the work thus:

$$x+6)\,x^2+11x+30\,($$

divide $x^2$, the first term of the dividend, by $x$, the first term of the divisor; the quotient is $x$. Multiply the *whole* divisor by $x$, and put the product $x^2+6x$ under the dividend. We then have

$$\begin{array}{l r}
 & x+6)\,x^2+11x+30\,(x \\
 & \underline{x^2+\ \,6x}\qquad\quad \\
\text{by subtraction} & 5x+30.\qquad
\end{array}$$

On repeating the process above explained we find that the next term in the quotient is $+5$.

The entire operation is more compactly written as follows:

$$\begin{array}{r}
x+6)\,x^2+11x+30\,(x+5 \\
\underline{x^2+\ \,6x}\qquad\qquad\quad \\
5x+30\qquad\quad \\
\underline{5x+30}\qquad\quad
\end{array}$$

The reason for the rule is this: the dividend may be divided into as many parts as may be convenient, and the complete quotient is found by taking the sum of all the partial quotients. Thus $x^2+11x+30$ is divided by the above process into two parts, namely $x^2+6x$, and $5x+30$, and each of these is divided by $x+6$; thus we obtain the complete quotient $x+5$.

*Example* 2. Divide $24x^2-65xy+21y^2$ by $8x-3y$.

$$\begin{array}{r}
8x-3y)\,24x^2-65xy+21y^2\,(3x-7y \\
\underline{24x^2-\ \,9xy}\qquad\qquad\qquad\quad \\
-56xy+21y^2\qquad\qquad\quad \\
\underline{-56xy+21y^2}\qquad\qquad\quad
\end{array}$$

## EXAMPLES VI. b.

Divide

1. $x^2+3x+2$ by $x+1$.
2. $x^2-7x+12$ by $x-3$.
3. $a^2-11a+30$ by $a-5$.
4. $a^2-49a+600$ by $a-25$.
5. $3x^2+10x+3$ by $x+3$.
6. $2x^2+11x+5$ by $2x+1$.
7. $5x^2+11x+2$ by $x+2$.
8. $2x^2+17x+21$ by $2x+3$.
9. $5x^2+16x+3$ by $x+3$.
10. $3x^2+34x+11$ by $3x+1$.
11. $4x^2+23x+15$ by $4x+3$.
12. $6x^2-7x-3$ by $2x-3$.

**13.** $3x^2+x-14$ by $x-2$.

**14.** $3x^2-x-14$ by $x+2$.

**15.** $6x^2-31x+35$ by $2x-7$.

**16.** $4x^2+x-14$ by $x+2$.

**17.** $12a^2-7ax-12x^2$ by $3a-4x$.

**18.** $15a^2+17ax-4x^2$ by $3a+4x$.

**19.** $12a^2-11ac-36c^2$ by $4a-9c$.

**20.** $9a^2+6ac-35c^2$ by $3a+7c$.

**21.** $60x^2-4xy-45y^2$ by $10x-9y$.

**22.** $-4xy-15y^2+96x^2$ by $12x-5y$.

**23.** $7x^3+96x^2-28x$ by $7x-2$.

**24.** $100x^3-3x-13x^2$ by $3+25x$.

51. In the examples given hitherto the divisor has been exactly contained in the dividend. It is a good exercise for the beginner to work cases in which the accurate value of the remainder is required.

*Example.* Find the remainder when $x^4-7x^3+9x^2-11x+15$ is divided by $x-5$.

$$\begin{array}{r@{}rrrrrl}
x-5) & x^4 & -7x^3 & +9x^2 & -11x & +15 & (x^3-2x^2-x-16 \\
 & x^4 & -5x^3 & & & & \\ \cline{2-3}
 & & -2x^3 & +9x^2 & & & \\
 & & -2x^3 & +10x^2 & & & \\ \cline{3-4}
 & & & -x^2 & -11x & & \\
 & & & -x^2 & +5x & & \\ \cline{4-5}
 & & & & -16x & +15 & \\
 & & & & -16x & +80 & \\ \cline{5-6}
 & & & & & -65. &
\end{array}$$

And the remainder is $-65$.

## EXAMPLES VI. c.

Find the remainder when

**1.** $x^3-6x^2+11x+2$ is divided by $x-2$.

**2.** $x^3-6x^2+12x-17$ .................. $x-3$.

**3.** $2x^3+5x^2-4x-7$ .................. $x+2$.

**4.** $3x^3-7x-9$ .................. $x+1$.

**5.** $4x^3+7x^2-3x-33$ .................. $4x-5$.

**6.** $27x^3+9x^2-3x-5$ .................. $3x-2$.

**7.** $16x^3-19+39x-46x^2$ .................. $8x-3$.

**8.** $8x-8x^2+5x^3+7$ .................. $5x-3$.

**9.** $21a^3-27a+15-26a^2$ is divided by $3a-5$.
**10.** $30x^4+11x^3-82x^2-5x+3$ .................. $2x-4+3x^2$.
**11.** $6x-5x^3+12x^4+20-33x^2$ .................. $x+4x^2-5$.
**12.** $30x+9-71x^3+28x^4-35x^2$ .................. $4x^2-13x+6$.

52. We add a few harder cases worked out in full.

*Example* 1. Divide $x^4+4a^4$ by $x^2+2xa+a^2$.

$$\begin{array}{l}
x^2+2xa+2a^2)\,x^4+4a^4\qquad\qquad(x^2-2xa+2a^2 \\
\qquad\qquad\qquad\;\; \underline{x^4+2x^3a+2x^2a^2} \\
\qquad\qquad\qquad\quad -2x^3a-2x^2a^2 \\
\qquad\qquad\qquad\quad \underline{-2x^3a-4x^2a^2-4xa^3} \\
\qquad\qquad\qquad\qquad\qquad\;\; 2x^2a^2+4xa^3+4a^4 \\
\qquad\qquad\qquad\qquad\qquad\;\; \underline{2x^2a^2+4xa^3+4a^4}
\end{array}$$

*Example* 2. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

$$\begin{array}{l}
a+b+c)\,a^3-3abc+b^3+c^3\,(a^2-ab-ac+b^2-bc+c^2 \\
\qquad\quad\;\; \underline{a^3+a^2b+a^2c} \\
\qquad\qquad -a^2b-a^2c-3abc \\
\qquad\qquad \underline{-a^2b-ab^2-abc} \\
\qquad\qquad\qquad -a^2c+ab^2-2abc \\
\qquad\qquad\qquad \underline{-a^2c-abc-ac^2} \\
\qquad\qquad\qquad\qquad\quad ab^2-abc+ac^2+b^3 \\
\qquad\qquad\qquad\qquad\quad \underline{ab^2+b^3+b^2c} \\
\qquad\qquad\qquad\qquad\qquad -abc+ac^2-b^2c \\
\qquad\qquad\qquad\qquad\qquad \underline{-abc-b^2c-bc^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad ac^2+bc^2+c^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{ac^2+bc^2+c^3}
\end{array}$$

**Note.** In the above example the dividend and successive remainders are arranged in *descending* powers of $a$.

The result of this division is very important and will be referred to later.

53. Sometimes it will be found convenient to arrange the expressions in *ascending* powers of some common letter.

*Example.* Divide $2a^3+10-16a-39a^2+15a^4$ by $2-4a-5a^2$.

$$\begin{array}{l}
2-4a-5a^2)\,10-16a-39a^2+2a^3+15a^4\,(5+2a-3a^2 \\
\qquad\qquad\quad \underline{10-20a-25a^2} \\
\qquad\qquad\qquad\quad 4a-14a^2+2a^3 \\
\qquad\qquad\qquad\quad \underline{4a-8a^2-10a^3} \\
\qquad\qquad\qquad\qquad -6a^2+12a^3+15a^4 \\
\qquad\qquad\qquad\qquad \underline{-6a^2+12a^3+15a^4}
\end{array}$$

54. When the coefficients are fractional the ordinary process may still be employed.

*Example.* Divide $\frac{1}{4}x^3+\frac{1}{72}xy^2+\frac{1}{12}y^3$ by $\frac{1}{2}x+\frac{1}{3}y$.

$$\begin{array}{l}
\frac{1}{2}x+\frac{1}{3}y\,\big)\,\frac{1}{4}x^3+\frac{1}{72}xy^2+\frac{1}{12}y^3\,\big(\,\frac{1}{2}x^2-\frac{1}{3}xy+\frac{1}{4}y^2 \\
\qquad\qquad\quad \underline{\frac{1}{4}x^3+\frac{1}{6}x^2y} \\
\qquad\qquad\qquad -\frac{1}{6}x^2y+\frac{1}{72}xy^2 \\
\qquad\qquad\qquad \underline{-\frac{1}{6}x^2y-\frac{1}{9}xy^2} \\
\qquad\qquad\qquad\qquad\qquad \frac{1}{8}xy^2+\frac{1}{12}y^3 \\
\qquad\qquad\qquad\qquad\qquad \underline{\frac{1}{8}xy^2+\frac{1}{12}y^3}
\end{array}$$

55. The following examples in division may be easily verified; they are of great importance and should be carefully noticed.

$$\text{I.}\quad\left\{\begin{array}{l}
\dfrac{x^2-y^2}{x-y}=x+y, \\[2ex]
\dfrac{x^3-y^3}{x-y}=x^2+xy+y^2, \\[2ex]
\dfrac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3,
\end{array}\right.$$

and so on; the divisor being $x-y$, the terms in the quotient *all positive*, and the index in the dividend *either odd or even.*

$$\text{II.}\quad\left\{\begin{array}{l}
\dfrac{x^3+y^3}{x+y}=x^2-xy+y^2, \\[2ex]
\dfrac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4, \\[2ex]
\dfrac{x^7+y^7}{x+y}=x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6,
\end{array}\right.$$

and so on; the divisor being $x+y$, the terms in the quotient *alternately positive and negative*, and the index in the dividend *always odd.*

$$\text{III.}\quad\left\{\begin{array}{l}
\dfrac{x^2-y^2}{x+y}=x-y, \\[2ex]
\dfrac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3, \\[2ex]
\dfrac{x^6-y^6}{x+y}=x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5,
\end{array}\right.$$

and so on; the divisor being $x+y$, the terms in the quotient *alternately positive and negative*, and the index in the dividend *always even.*

IV. The expressions $x^2+y^2$, $x^4+y^4$, $x^6+y^6$... (where the index is *even*, and the terms *both positive*) are *never* divisible by $x+y$ or $x-y$.

All these different cases may be more concisely stated as follows:

(1) $x^n-y^n$ is divisible by $x-y$ if $n$ be *any* whole number.

(2) $x^n+y^n$ is divisible by $x+y$ if $n$ be any *odd* whole number.

(3) $x^n-y^n$ is divisible by $x+y$ if $n$ be any *even* whole number.

(4) $x^n+y^n$ is never divisible by $x+y$ or $x-y$, when $n$ is an *even* whole number.

## EXAMPLES VI. d.

Divide

1. $14x^4+45x^3y+78x^2y^2+45xy^3+14y^4$ by $2x^2+5xy+7y^2$.
2. $x^5-5x^4+9x^3-6x^2-x+2$ by $x^2-3x+2$.
3. $x^5-4x^4+3x^3+3x^2-3x+2$ by $x^2-x-2$.
4. $x^6-x^4y+x^3y^2-x^3-y^3$ by $x^3-x-y$.
5. $x^5+x^4y-x^3y^2+x^3-2xy^2+y^3$ by $x^2+xy-y^2$.
6. $x^5-2x^4-4x^3+19x^2-31x+15$ by $x^3-7x+5$.
7. $2x^3-8x+x^4+12-7x^2$ by $x^2+2-3x$.
8. $14a^4-45a^3b+78a^2b^2-45ab^3+14b^4$ by $2a^2-5ab+7b^2$.
9. $4m^3+9m^2+m^5+2m^4-31m+15$ by $m^2+2m-3$.
10. $a^3+b^3+3abc-c^3$ by $a+b-c$.
11. $8x^3-4x^2-128x+x^4-192$ by $x^2-16$.
12. $a^4+6a^3+13a^2+12a+4$ by $a^2+3a+2$.
13. $a^9-b^9$ by $a^3-b^3$.
14. $x^9-y^9$ by $x^2+xy+y^2$.
15. $x^7-2y^{14}-7x^5y^4-7xy^{12}+14x^3y^8$ by $x-2y^2$.
16. $a^3+3a^2b+b^3-1+3ab^2$ by $a+b-1$.
17. $x^8-y^8$ by $x^3+x^2y+xy^2+y^3$.
18. $a^{12}-b^{12}$ by $a^2-b^2$.
19. $a^{12}+2a^6b^6+b^{12}$ by $a^4+2a^2b^2+b^4$.
20. $b^6+c^6+6b^5c+6bc^5+15b^4c^2+15b^2c^4+20b^3c^3$ by $b+c$.

Find the quotient of

21. $\frac{1}{8}a^3-\frac{9}{4}a^2x+\frac{27}{2}ax^2-27x^3$ by $\frac{1}{2}a-3x$.
22. $\frac{1}{27}a^3-\frac{1}{12}a^2+\frac{1}{16}a-\frac{1}{64}$ by $\frac{1}{3}a-\frac{1}{4}$.
23. $\frac{3}{4}a^2c^3+\frac{6}{125}a^5$ by $\frac{1}{5}a^2+\frac{1}{2}ac$.
24. $\frac{9}{16}a^4-\frac{3}{4}a^3-\frac{7}{4}a^2+\frac{4}{3}a+\frac{16}{9}$ by $\frac{3}{2}a^2-\frac{8}{3}-a$.
25. $36x^2+\frac{1}{9}y^2+\frac{1}{4}-4xy-6x+\frac{1}{3}y$ by $6x-\frac{1}{3}y-\frac{1}{2}$.
26. $\frac{8}{27}a^5-\frac{243}{512}ax^4$ by $\frac{2}{3}a-\frac{3}{4}x$.

# CHAPTER VII.

## REMOVAL AND INSERTION OF BRACKETS.

56. WE frequently find it necessary to enclose within brackets part of an expression already enclosed within brackets. For this purpose it is usual to employ brackets of different forms. The brackets in common use are ( ), { }, [ ]. Sometimes a line called a "vinculum" is drawn over the symbols to be connected; thus $a-\overline{b+c}$ is used with the same meaning as $a-(b+c)$, and hence $a-\overline{b+c}=a-b-c$.

57. To remove brackets it is usually best to begin with the inside pair, and in dealing with each pair in succession we apply the rules already given in Arts. 21, 22.

*Example* 1. Simplify, by removing brackets, the expression

$$a-2b-[4a-6b-\{3a-c+(5a-2b-\overline{3a-c+2b})\}].$$

Removing the brackets one by one, we have

$$\begin{aligned}&a-2b-[4a-6b-\{3a-c+(5a-2b-3a+c-2b)\}]\\
&=a-2b-[4a-6b-\{3a-c+\ 5a-2b-3a+c-2b\}]\\
&=a-2b-[4a-6b-\ 3a+c-\ 5a+2b+3a-c+2b]\\
&=a-2b-\ 4a+6b+\ 3a-c+\ 5a-2b-3a+c-2b\\
&=2a, \text{ by collecting like terms.}\end{aligned}$$

*Example* 2. Simplify the expression

$$-[-2x-\{3y-(2x-3y)+(3x-2y)\}+2x].$$

$$\begin{aligned}\text{The expression}&=-[-2x-\{3y-2x+3y+3x-2y\}+2x]\\
&=-[-2x-3y+2x-3y-3x+2y+2x]\\
&=2x+3y-2x+3y+3x-2y-2x\\
&=x+4y.\end{aligned}$$

## EXAMPLES VII. a.

Simplify by removing brackets

1. $a-(b-c)+a+(b-c)+b-(c+a)$.
2. $a-[b+\{a-(b+a)\}]$.
3. $a-[2a-\{3b-(4c-2a)\}]$.
4. $\{a-(b-c)\}+\{b-(c-a)\}-\{c-(a-b)\}$.
5. $2a-(5b+[3c-a])-(5a-[b+c])$.
6. $-\{-[-(a-\overline{b-c})]\}$.
7. $-[a-\{b-(c-a)\}]-[b-\{c-(a-b)\}]$.
8. $-(-(-(-x)))-(-(-y))$.
9. $-[-\{-(b+c-a)\}]+[-\{-(c+a-b)\}]$.
10. $-5x-[3y-\{2x-(2y-x)\}]$.
11. $-(-(-a))-(-(-(-x)))$.
12. $3a-[a+b-\{a+b+c-(a+b+c+d)\}]$.
13. $-2a-[3x+\{3c-(4y+3x+2a)\}]$.
14. $3x-[5y-\{6z-(4x-7y)\}]$.
15. $-[5x-(11y-3x)]-[5y-(3x-6y)]$.
16. $-[15x-\{14y-(15z+12y)-(10x-15z)\}]$.
17. $8x-\{16y-[3x-(12y-x)-8y]+x\}$.
18. $-[x-\{z+(x-z)-(z-x)-z\}-x]$.
19. $-[a+\{a-(a-x)-(a+x)-a\}-a]$.
20. $-[a-\{a+(x-a)-(x-a)-a\}-2a]$.

58. A coefficient placed before any bracket indicates that every term of the expression within the bracket is to be multiplied by that coefficient.

**Note.** The line between the numerator and denominator of a fraction is a kind of vinculum. Thus $\frac{x-5}{3}$ is equivalent to $\frac{1}{3}(x-5)$.

Again, an expression of the form $\sqrt{(x+y)}$ is often written $\sqrt{x+y}$, the line above being regarded as a vinculum indicating the square root of the compound expression $x+y$ *taken as a whole.*

Thus $\sqrt{25+144}=\sqrt{169}=13$,

whereas $\sqrt{25}+\sqrt{144}=5+12=17$.

59. Sometimes it is advisable to simplify in the course of the work.

*Example.* Find the value of

$$84 - 7[-11x - 4\{-17x + 3(8 - \overline{9 - 5x})\}].$$

$$\begin{aligned}
\text{The expression} &= 84 - 7[-11x - 4\{-17x + 3(8 - 9 + 5x)\}] \\
&= 84 - 7[-11x - 4\{-17x + 3(5x - 1)\}] \\
&= 84 - 7[-11x - 4\{-17x + 15x - 3\}] \\
&= 84 - 7[-11x - 4\{-2x - 3\}] \\
&= 84 - 7[-11x + 8x + 12] \\
&= 84 - 7[-3x + 12] \\
&= 84 + 21x - 84 \\
&= 21x.
\end{aligned}$$

When the beginner has had a little practice the number of steps may be considerably diminished.

## EXAMPLES VII. b.

Simplify by removing brackets

1. $a - [2b + \{3c - 3a - (a + b)\} + 2a - (b + 3c)]$.
2. $a + b - (c + a - [b + c - (a + b - \{c + a - (b + c - a)\})])$.
3. $a - (b - c) - [a - b - c - 2\{b + c - 3(c - a) - d\}]$.
4. $2x - (3y - 4z) - \{2x - (3y + 4z)\} - \{3y - (4z + 2x)\}$.
5. $b + c - (a + b - [c + a - (b + c - \{a + b - (c + a - b)\})])$.
6. $3b - \{5a - [6a + 2(10a - b)]\}$.
7. $a - (b - c) - [a - b - c - 2\{b + c\}]$.
8. $3a^2 - [6a^2 - \{8b^2 - (9c^2 - 2a^2)\}]$.
9. $b - (c - a) - [b - a - c - 2\{c + a - 3(a - b) - d\}]$.
10. $-20(a - d) + 3(b - c) - 2[b + c + d - 3\{c + d - 4(d - a)\}]$.
11. $-4(a + d) + 24(b - c) - 2[c + d + a - 3\{d + a - 4(b + c)\}]$.
12. $-10(a + b) - [c + a + b - 3\{a + 2b - (c + a - b)\}] + 4c$.
13. $a - 2(b - c) - [-\{-(4a - b - c - 2\{a + b + c\})\}]$.
14. $8(b - c) - [-\{a - b - 3(c - b + a)\}]$.
15. $2(3b - 5a) - 7[a - 6\{2 - 5(a - b)\}]$.
16. $6\{a - 2[b - 3(c + d)]\} - 4\{a - 3[b - 4(c - d)]\}$.
17. $5\{a - 2[a - 2(a + x)]\} - 4\{a - 2[a - 2(a + x)]\}$.
18. $-10\{a - 6[a - (b - c)]\} + 60\{b - (c + a)\}$.

19. $-3\{-2[-4(-a)]\}+5\{-2[-2(-a)]\}$.

20. $-2\{-[-(x-y)]\}+\{-2[-(x-y)]\}$.

21. $\frac{1}{4}\{a-5(b-a)\}-\frac{3}{2}\left\{\frac{1}{3}\left(b-\frac{a}{3}\right)-\frac{2}{9}\left[a-\frac{3}{4}\left(b-\frac{4a}{5}\right)\right]\right\}$.

22. $35\left[\frac{3x-4y}{5}-\frac{1}{10}\{3x-\frac{5}{7}(7x-4y)\}\right]+8(y-2x)$.

23. $\frac{3}{8}\{\frac{4}{3}(a-b)-8(b-c)\}-\left\{\frac{b-c}{2}-\frac{c-a}{3}\right\}-\frac{1}{2}\{c-a-\frac{2}{3}(a-b)\}$.

24. $\frac{1}{2}x-\frac{1}{2}(\frac{2}{3}y-\frac{1}{2}z)-[x-\{\frac{1}{2}x-(\frac{1}{3}y-\frac{1}{4}z)\}-(\frac{2}{3}y-\frac{1}{2}z)]$.

## INSERTION OF BRACKETS.

60. The converse operation of inserting brackets is important. The rules for doing this have been enunciated in Arts. 21, 22; for convenience we repeat them.

1. Any part of an expression may be enclosed within brackets and the sign + prefixed, the sign of every term within the brackets remaining unaltered.

2. Any part of an expression may be enclosed within brackets and the sign − prefixed, provided the sign of every term within the brackets be changed.

*Examples.* $a-b+c-d-e=a-b+(c-d-e)$.

$a-b+c-d-e=a-(b-c)-(d+e)$.

$x^2-ax+bx-ab=(x^2-ax)+(bx-ab)$.

$xy-ax-by+ab=(xy-by)-(ax-ab)$.

61 The terms of an expression can be bracketed in various ways.

*Example.* The expression $ax-bx+cx-ay+by-cy$

may be written $(ax-bx)+(cx-ay)+(by-cy)$,

or $(ax-bx+cx)-(ay-by+cy)$,

or $(ax-ay)-(bx-by)+(cx-cy)$.

62. Whenever a factor is common to every term within a bracket, it may be removed and placed outside as a multiplier of the expression within the bracket.

*Example* 1. In the expression

$$ax^3 - cx + 7 - dx^2 + bx - c - dx^3 + bx^2 - 2x$$

bracket together the powers of $x$ so as to have the sign + before each bracket.

$$\begin{aligned}\text{The expression} &= (ax^3 - dx^3) + (bx^2 - dx^2) + (bx - cx - 2x) + (7 - c)\\ &= x^3(a - d) + x^2(b - d) + x(b - c - 2) + (7 - c)\\ &= (a - d)x^3 + (b - d)x^2 + (b - c - 2)x + 7 - c.\end{aligned}$$

In this last result the compound expressions $a - d$, $b - d$, $b - c - 2$ are regarded as the coefficients of $x^3$, $x^2$ and $x$ respectively.

*Example* 2. In the expression $-a^2x - 7a + a^2y + 3 - 2x - ab$ bracket together the powers of $a$ so as to have the sign $-$ before each bracket.

$$\begin{aligned}\text{The expression} &= -(a^2x - a^2y) - (7a + ab) - (2x - 3)\\ &= -a^2(x - y) - a(7 + b) - (2x - 3)\\ &= -(x - y)a^2 - (7 + b)a - (2x - 3).\end{aligned}$$

## EXAMPLES VII. c.

In the following expressions bracket the powers of $x$ so that the signs before all the brackets shall be positive:

1. $ax^4 + bx^2 + 5 + 2bx - 5x^2 + 2x^4 - 3x.$
2. $3bx^2 - 7 - 2x + ab + 5ax^3 + cx - 4x^2 - bx^3.$
3. $2 - 7x^3 + 5ax^2 - 2cx + 9ax^3 + 7x - 3x^2.$
4. $2cx^5 - 3abx + 4dx - 3bx^4 - a^2x^5 + x^4.$

In the following expressions bracket the powers of $x$ so that the signs before all the brackets shall be negative:

5. $ax^2 + 5x^3 - a^2x^4 - 2bx^3 - 3x^2 - bx^4.$
6. $7x^3 - 3c^2x - abx^5 + 5ax + 7x^5 - abcx^3.$
7. $ax^2 + a^2x^3 - bx^2 - 5x^2 - cx^3.$
8. $3b^2x^4 - bx - ax^4 - cx^4 - 5c^2x - 7x^4.$

Simplify the following expressions, and in each result regroup the terms according to powers of $x$:

9. $ax^3 - 2cx - [bx^2 - \{cx - dx - (bx^3 + 3cx^2)\} - (cx^2 - bx)].$
10. $5ax^3 - 7(bx - cx^2) - \{6bx^2 - (3ax^2 + 2ax) - 4cx^3\}.$
11. $ax^2 - 3\{-ax^3 + 3bx - 4[\frac{1}{6}cx^3 - \frac{2}{3}(ax - bx^2)]\}.$
12. $x^5 - 4bx^4 - \dfrac{1}{6}\left[12ax - 4\left\{3bx^4 - 9\left(\dfrac{cx}{2} - bx^5\right) - \frac{3}{2}ax^4\right\}\right].$
13. $x\{x - b - x(a - bx)\} + ax - x\{x - x(ax - b)\}.$

63. In certain cases of addition, multiplication, &c., of expressions which involve literal coefficients, the results may be more conveniently written by grouping the terms according to powers of some common letter.

*Example* 1. Add together $ax^3 - 2bx^2 + 3$, $bx - cx^3 - x^2$ and $x^3 - ax^2 + cx$.

$$\begin{aligned}\text{The sum} &= ax^3 - 2bx^2 + 3 + bx - cx^3 - x^2 + x^3 - ax^2 + cx\\ &= ax^3 - cx^3 + x^3 - ax^2 - 2bx^2 - x^2 + bx + cx + 3\\ &= (a - c + 1)\, x^3 - (a + 2b + 1)\, x^2 + (b + c)\, x + 3.\end{aligned}$$

*Example* 2. Multiply $ax^2 - 2bx + 3c$ by $px - q$.

$$\begin{aligned}\text{The product} &= (ax^2 - 2bx + 3c)\,(px - q)\\ &= apx^3 - 2bpx^2 + 3cpx - aqx^2 + 2bqx - 3cq\\ &= apx^3 - (2bp + aq)\, x^2 + (3cp + 2bq)\, x - 3cq.\end{aligned}$$

## EXAMPLES VII. d.

Add together the following expressions, and in each case arrange the result according to powers of $x$:

1. $ax^3 - 2cx$, $bx^2 - cx^3$ and $cx^2 - x$.
2. $x^2 - x - 1$, $ax^2 - bx^3$, $bx + x^3$.
3. $a^2x^3 - 5x$, $2ax^2 - 5ax^3$, $2x^3 - bx^2 - ax$.
4. $ax^2 + bx - c$, $qx - r - px^2$, $x^2 + 2x + 3$.
5. $px^3 - qx$, $qx^2 - px$, $q - x^3$, $px^2 + qx^3$.

Multiply together the following expressions, and in each case arrange the result according to powers of $x$:

6. $ax^2 + bx + 1$ and $cx + 2$.
7. $cx^2 - 2x + 3$, and $ax - b$.
8. $ax^2 - bx - c$ and $px + q$.
9. $2x^2 - 3x - 1$ and $bx + c$.
10. $ax^2 - 2bx + 3c$ and $x - 1$.
11. $px^2 - 2x - q$ and $ax - 3$.
12. $x^3 + ax^2 - bx - c$ and $x^3 - ax^2 - bx + c$.
13. $ax^3 - x^2 + 3x - b$ and $ax^3 + x^2 + 3x + b$.
14. $x^4 - ax^3 - bx^2 + cx + d$ and $x^4 + ax^3 - bx^2 - cx + d$.

# CHAPTER VIII.

## SIMPLE EQUATIONS.

64. AN **equation** asserts that two expressions are equal, but we do not usually employ the word equation in so wide a sense.

Thus the statement $x+3+x+4=2x+7$, which is *always* true whatever value $x$ may have, is called an **identical equation**, or briefly an **identity.**

The parts of an equation to the right and left of the sign of equality are called **members** or **sides** of the equation, and are distinguished as the *right side* and *left side.*

65. Certain equations are only true for particular values of the symbols employed. Thus $3x=6$ is only true when $x=2$, and is called an **equation of condition,** or more usually an equation. Consequently an *identity* is an equation which is always true whatever be the values of the symbols involved; whereas an *equation* (in the ordinary use of the word) is only true for particular values of the symbols. In the above example $3x=6$, the value 2 is said to **satisfy** the equation. The object of the present chapter is to explain how to treat an equation of the simplest kind in order to discover the value which satisfies it.

66. The letter whose value it is required to find is called the **unknown quantity.** The process of finding its value is called **solving the equation.** The value so found is called the **root** or the **solution** of the equation.

67. The solution of equations, and the operations subsidiary to it, form an extremely important part of Mathematics. All sorts of mathematical problems consist in the indirect determination of some quantity by means of its relations to other quantities which are known, and these relations are all expressed by means of equations. The operation in general of solving a problem in Mathematics, other than a transformation, is firstly to express the conditions of the problem by means of one or more equations, and secondly to solve these equations. For example, the problem which is expressed by the equation above given is the very simple question, "What is the number such that if multiplied by 3, the product is 6?" In the present chapter, it is the second of these two operations, the solution of an equation, that is considered.

68. An equation which involves the unknown quantity in the first degree is called a **simple equation.** It is usual to denote the unknown quantity by the letter $x$.

69. The process of solving a simple equation depends only upon the following axioms:

1. If to equals we add equals the sums are equal.
2. If from equals we take equals the remainders are equal.
3. If equals are multiplied by equals the products are equal.
4. If equals are divided by equals the quotients are equal.

70. Consider the equation $7x=14$.

It is required to find what numerical value $x$ must have consistent with this statement.

Dividing both sides by 7 we get

$$x=2 \quad \text{.......................(Axiom 4).}$$

Similarly, if $\dfrac{x}{2}=-6$,

multiplying both sides by 2, we get

$$x=-12 \quad \text{....................(Axiom 3).}$$

Again, in the equation $7x-2x-x=23+15-10$, by collecting terms, we have $4x=28$.

$$\therefore x=7.$$

71. To solve $3x-8=x+12$.

This case differs from the preceding in that the unknown quantity occurs on both sides of the equation. We can, however, **transpose** any term from one side to the other by simply *changing its sign.* This we proceed to shew.

Subtract $x$ from both sides of the equation, and we get

$$3x-x-8=12 \text{.......................(Axiom 2).}$$

Adding 8 to both sides, we have

$$3x-x=12+8 \quad \text{.................(Axiom 1).}$$

Thus we see that $+x$ has been removed from one side, and appears as $-x$ on the other; and $-8$ has been removed from one side and appears as $+8$ on the other.

It is evident that similar steps may be employed in all cases. Hence we may enunciate the following rule:

RULE. *Any term may be transposed from one side of the equation to the other by changing its sign.*

It appears from this that *we may change the sign of every term in an equation;* for this is equivalent to transposing all the terms, and then making the right and left hand members change places.

*Example.* Take the equation $-3x-12=x-24$.

Transposing, $-x+24=3x+12$,

or $3x+12=-x+24$,

which is the original equation with the sign of every term changed.

72. To solve $\frac{x}{2}-3=\frac{x}{4}+\frac{x}{5}$.

Here it will be convenient to begin by clearing the equation of *fractional* coefficients. This can always be done by multiplying both sides of the equation by the least common multiple of the denominators.

Thus, multiplying by 20,

$$10x-60=5x+4x;$$

transposing, $10x-5x-4x=60$;

$$\therefore x=60.$$

73. We can now give a general rule for solving any simple equation with one unknown quantity.

RULE. *First, if necessary, clear of fractions; then transpose all the terms containing the unknown quantity to one side of the equation, and the known quantities to the other. Collect the terms on each side; divide both sides by the coefficient of the unknown quantity, and the value required is obtained.*

*Example* 1. Solve $5(x-3)-7(6-x)+3=24-3(8-x)$.

Removing brackets, $5x-15-42+7x+3=24-24+3x$;

transposing, $5x+7x-3x=24-24+15+42-3$;

$$\therefore 9x=54;$$
$$\therefore x=6.$$

*Example* 2. Solve $5x-(4x-7)(3x-5)=6-3(4x-9)(x-1)$.

Simplifying, we have

$$5x-(12x^2-41x+35)=6-3(4x^2-13x+9).$$

And by removing brackets

$$5x-12x^2+41x-35=6-12x^2+39x-27.$$

Erase the term $-12x^2$ on each side and transpose;

thus $5x+41x-39x=6-27+35$;

$$\therefore 7x=14;$$
$$\therefore x=2.$$

NOTE. Since the – sign before a bracket affects every term within it, in the first line of work of Ex. 2, we do not remove the brackets until we have formed the products.

*Example* 3. Solve $7x-5[x-\{7-6(x-3)\}]=3x+1$.

Removing brackets, we have

$$7x-5[x-\{7-6x+18\}]=3x+1,$$
$$7x-5[x-25+6x]=3x+1,$$
$$7x-5x+125-30x=3x+1;$$

transposing, $$7x-5x-30x-3x=1-125;$$
$$\therefore\quad -31x=-124;$$
$$\therefore\quad x=4.$$

74. It is extremely useful for the beginner to acquire the habit of occasionally verifying, that is, proving the truth of his results. Proofs of this kind are interesting and convincing; and the habit of applying such tests tends to make the student self-reliant and confident in his own accuracy.

In the case of simple equations we have only to shew that when we substitute the value of $x$ in the two sides of the equation we obtain the same result.

*Example.* To shew that $x=2$ satisfies the equation

$5x-(4x-7)(3x-5)=6-3(4x-9)(x-1)$.........Ex. 2. Art. 73.

When $x=2$, the left side $5x-(4x-7)(3x-5)=10-(8-7)(6-5)$
$$=10-1$$
$$=9.$$

The right side $6-3(4x-9)(x-1)=6-3(8-9)(2-1)$
$$=6-3(-1)$$
$$=9.$$

Thus, since these two results are the same, $x=2$ satisfies the equation.

## EXAMPLES VIII. a.

Solve the following equations:

1. $3x+15=x+25$.
2. $2x-3=3x-7$.
3. $3x+4=5(x-2)$.
4. $2x+3=16-(2x-3)$.
5. $8(x-1)+17(x-3)=4(4x-9)+4$.
6. $15(x-1)+4(x+3)=2(7+x)$.

7. $5x-6(x-5)=2(x+5)+5(x-4)$.
8. $8(x-3)-(6-2x)=2(x+2)-5(5-x)$.
9. $7(25-x)-2x=2(3x-25)$.
10. $3(169-x)-(78+x)=29x$.
11. $5x-17+3x-5=6x-7-8x+115$.
12. $7x-39-10x+15=100-33x+26$.
13. $118-65x-123=15x+35-120x$.
14. $157-21(x+3)=163-15(2x-5)$.
15. $179-18(x-10)=158-3(x-17)$.
16. $97-5(x+20)=111-8(x+3)$.
17. $x-[3+\{x-(3+x)\}]=5$.
18. $5x-(3x-7)-\{4-2x-(6x-3)\}=10$.
19. $14x-(5x-9)-\{4-3x-(2x-3)\}=30$.
20. $25x-19-[3-\{4x-5\}]=3x-(6x-5)$.
21. $(x+1)(2x+1)=(x+3)(2x+3)-14$.
22. $(x+1)^2-(x^2-1)=x(2x+1)-2(x+2)(x+1)+20$.
23. $2(x+1)(x+3)+8=(2x+1)(x+5)$.
24. $6(x^2-3x+2)-2(x^2-1)=4(x+1)(x+2)-24$.
25. $2(x-4)-(x^2+x-20)=4x^2-(5x+3)(x-4)-64$.
26. $(x+15)(x-3)-(x^2-6x+9)=30-15(x-1)$.
27. $2x-5\{3x-7(4x-9)\}=66$.
28. $20(2-x)+3(x-7)-2[x+9-3\{9-4(2-x)\}]=22$.
29. $x+2-[x-8-2\{8-3(5-x)-x\}]=0$.
30. $3(5-6x)-5[x-5\{1-3(x-5)\}]=23$.
31. $(x+1)(2x+3)=2(x+1)^2+8$.
32. $3(x-1)^2-3(x^2-1)=x-15$.
33. $(3x+1)(2x-7)=6(x-3)^2+7$.
34. $x^2-8x+25=x(x-4)-25(x-5)-16$.
35. $x(x+1)+(x+1)(x+2)=(x+2)(x+3)+x(x+4)-9$.
36. $2(x+2)(x-4)=x(2x+1)-21$.
37. $(x+1)^2+2(x+3)^2=3x(x+2)+35$.
38. $4(x+5)^2-(2x+1)^2=3(x-5)+180$.
39. $84+(x+4)(x-3)(x+5)=(x+1)(x+2)(x+3)$.
40. $(x+1)(x+2)(x+6)=x^3+9x^2+4(7x-1)$.

75. The following examples illustrate the most useful methods of solving fractional equations.

*Example* 1. Solve $4-\frac{x-9}{8}=\frac{x}{22}-\frac{1}{2}$.

Multiply by 88, which is the least common multiple of the denominators, and we get

$$352-11(x-9)=4x-44;$$

removing brackets, $352-11x+99=4x-44$;

transposing, $-11x-4x=-44-352-99$;

collecting terms and changing signs, $15x=495$;

$$\therefore\ x=33.$$

NOTE. In this equation $-\frac{x-9}{8}$ is regarded as a single term with the minus sign before it. In fact it is equivalent to $-\frac{1}{8}(x-9)$, the line between the numerator and denominator having the same effect as a bracket.

76. In certain cases it will be found more convenient not to multiply throughout by the L.C.M. of the denominator, but to clear of fractions in two or more steps.

*Example* 2. Solve $\frac{x-4}{3}+\frac{2x-3}{35}=\frac{5x-32}{9}-\frac{x+9}{28}$.

First multiply throughout by 9, and we have

$$3x-12+\frac{18x-27}{35}=5x-32-\frac{9x+81}{28};$$

transposing, $\frac{18x-27}{35}+\frac{9x+81}{28}=2x-20$.

Now clear of fractions by multiplying by $5\times7\times4$ or 140, and we have

$$72x-108+45x+405=280x-2800;$$
$$\therefore\ 2800-108+405=280x-72x-45x;$$
$$\therefore\ 3097=163x;$$
$$\therefore\ x=19.$$

77. To solve equations whose coefficients are decimals, we may express the decimals as vulgar fractions, and proceed as before; but it is often found more simple to work entirely in decimals.

*Example* 1. Solve $\cdot\dot{6}x+\cdot25-\frac{1}{9}x=1\cdot\dot{8}-\cdot75x-\frac{1}{3}$.

Expressing the decimals as vulgar fractions, we have

$$\tfrac{2}{3}x+\tfrac{1}{4}-\tfrac{1}{9}x=1\tfrac{8}{9}-\tfrac{3}{4}x-\tfrac{1}{3};$$

clearing of fractions, $24x+9-4x=68-27x-12$;

transposing, $24x-4x+27x=68-12-9$,

$$47x=47;$$
$$\therefore\ x=1.$$

*Example* 2. Solve $\cdot375x-1\cdot875=\cdot12x+1\cdot185$.

transposing, $\cdot375x-\cdot12x=1\cdot185+1\cdot875$;

collecting terms, $(\cdot375-\cdot12)\,x=3\cdot06$,

that is, $\cdot255x=3\cdot06$;

$$\therefore\ x=\frac{3\cdot06}{\cdot255}$$

$$=12.$$

## EXAMPLES VIII. b.

1. $\dfrac{x}{4}+\dfrac{x-5}{3}=10.$

2. $\dfrac{x-5}{10}+\dfrac{x+5}{5}=5.$

3. $\dfrac{x-2}{2}+\dfrac{x+10}{9}=5.$

4. $\dfrac{x+19}{5}=3+\dfrac{x}{4}.$

5. $\dfrac{x-4}{7}=\dfrac{x-10}{5}.$

6. $\dfrac{x-1}{8}=1+\dfrac{x+1}{18}.$

7. $\dfrac{4(x+2)}{5}=7+\dfrac{5x}{13}.$

8. $\dfrac{x+4}{14}+\dfrac{x-4}{6}=2.$

9. $\dfrac{x+20}{9}+\dfrac{3x}{7}=6.$

10. $\dfrac{x-8}{7}+\dfrac{x-3}{3}+\dfrac{5}{21}=0.$

11. $\dfrac{x+5}{6}-\dfrac{x+1}{9}=\dfrac{x+3}{4}.$

12. $\dfrac{4-5x}{6}-\dfrac{1-2x}{3}=\dfrac{13}{42}.$

13. $\dfrac{5(x+5)}{8}-\dfrac{2(x-3)}{7}=5\tfrac{19}{28}.$

14. $\dfrac{4(x+2)}{3}-\dfrac{6(x-7)}{7}=12.$

15. $1+\dfrac{x}{2}-\dfrac{2x}{3}=\dfrac{3x}{4}-4\tfrac{1}{2}.$

16. $\dfrac{x}{2}+\dfrac{x}{3}-\dfrac{x}{4}+\dfrac{x}{5}=7\tfrac{5}{6}.$

17. $\dfrac{3}{16}(x-1)-\dfrac{5}{12}(x-4)=\dfrac{2}{5}(x-6)+\dfrac{5}{48}.$

18. $x+\dfrac{5}{3}(x-7)-\dfrac{6}{7}(x-8)=3x-14\tfrac{1}{3}.$

19. $\dfrac{3x}{4}-\dfrac{6}{17}(x+10)-(x-3)=\dfrac{x-7}{51}-4\tfrac{3}{4}.$

20. $\dfrac{7x}{5}-\dfrac{1}{14}(x-11)=\dfrac{3}{7}(x-25)+34.$

21. $3+\dfrac{x}{4}=\dfrac{1}{2}\left(4-\dfrac{x}{3}\right)-\dfrac{5}{6}+\dfrac{1}{3}\left(11-\dfrac{x}{2}\right).$

**22.** $\frac{1}{5}(x-8)+\frac{4+x}{4}+\frac{x-1}{7}=7-\frac{23-x}{5}$.

**23.** $\frac{1}{3}\left(\frac{x}{4}-3\right)+\frac{5x}{6}-\frac{5x}{4}=\frac{x-12}{5}-\frac{x+3}{3}$.

**24.** $x-\left(3x-\frac{2x-5}{10}\right)=\frac{1}{6}(2x-57)-\frac{5}{3}$.

**25.** $\frac{x}{4}-\frac{x+10}{5}+4\frac{3}{4}=x-1-\frac{x-2}{3}$.

**26.** $\cdot 5x-\cdot\dot{3}x=\cdot 25x-1$. **27.** $3+\frac{x}{\cdot 5}=7-\frac{x}{\cdot 2}$.

**28.** $2\cdot 25x-\cdot 125=3x+3\cdot 75$. **29.** $\cdot 2x-\cdot 1\dot{6}x=\cdot 6-\cdot\dot{3}$.

**30.** $\cdot 6x-\cdot 7x+\cdot 75x-\cdot 875x+15=0$.

**31.** $12\{3x-\cdot 25(x-4)-\cdot\dot{3}(5x+14)\}=47$.

**32.** $\frac{\cdot 25(x-3)+\cdot\dot{3}(x-4)}{\cdot 125}=5x-19$.

**33.** $\frac{x+\cdot 75}{\cdot 125}-\frac{x-\cdot 25}{\cdot 25}=15$. **34.** $\frac{5}{6}x+\cdot 25x-\cdot\dot{3}x=x-3$.

**35.** $\cdot 5x-\cdot\dot{2}x=\cdot\dot{3}x-1\cdot 5$. **36.** $1\cdot 5=\frac{\cdot 36}{\cdot 2}-\frac{\cdot 09x-\cdot 18}{\cdot 9}$.

78. Before concluding this chapter it will be worth while to draw attention to the following cases which occur so frequently in solving equations that the beginner should learn to write down the solution at sight.

*Case I.* Suppose $\frac{7x}{5}=\frac{4}{3}$.

Multiplying both sides by 5, we have

$$\left.\begin{aligned} 7x&=\frac{4\times 5}{3} \\ \therefore\ x&=\frac{4\times 5}{3\times 7} \end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

*Case II.* Suppose $\frac{5}{3x}=\frac{9}{7}$.

Multiplying both sides by $3x$, we have

$$\left.\begin{aligned} 5&=\frac{9\times 3x}{7} \\ 5\times 7&=9\times 3x \\ \therefore\ \frac{5\times 7}{9\times 3}&=x \end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

By a careful examination of the results in (1) and (2), the truth of the following principles will be evident:

*Any factor of the numerator of one side of an equation may be transferred to the denominator of the other side, and any factor of the denominator of one side may be transferred to the numerator of the other side.*

The ready application of these principles will be found very useful.

*Example* 1. If $$\frac{3x}{14}=\frac{9}{35},$$

then $$x=\frac{9\times 14}{35\times 3}=1\tfrac{1}{5}.$$

*Example* 2. If $$\frac{2}{x}=-5,$$

then $$\frac{2}{5}=-x; \qquad \therefore\ x=-\frac{2}{5}.$$

After a little practice the arithmetic should be performed mentally, and the intermediate steps omitted.

## EXAMPLES VIII. c.

Write down the values of $x$ which satisfy the following equations:

1. $\frac{2}{x}=\frac{3}{4}$.
2. $\frac{3}{7}=\frac{x}{14}$.
3. $\frac{3}{5}=\frac{6}{x}$.
4. $-\frac{x}{3}=2$.
5. $\frac{x}{17}=\frac{29}{51}$.
6. $\frac{2x}{15}=\frac{8}{45}$
7. $\frac{3}{2x}=-\frac{1}{8}$.
8. $-\frac{4x}{3}=-\frac{1}{2}$.
9. $\frac{5}{3x}=\frac{25}{27}$.
10. $\frac{5}{2x}=-\frac{10}{3}$.
11. $\frac{13}{21}=\frac{65x}{84}$.
12. $-\frac{7}{2}=\frac{1}{3x}$.
13. $\frac{3}{8}=\frac{x}{4}$.
14. $\frac{8}{21x}=-\frac{4}{7}$.
15. $\frac{36}{35}=\frac{9}{5x}$.
16. $\frac{5}{8}=\frac{15}{2x}$.
17. $\frac{x}{18}=\frac{9}{42}$.
18. $\frac{4}{3x}=\frac{16}{27}$.
19. $\frac{49}{15}=\frac{7}{3x}$.
20. $\frac{56}{15}=\frac{8x}{3}$.
21. $\frac{19x}{7}=\frac{57}{49}$.

# CHAPTER IX.

## SYMBOLICAL EXPRESSION.

79. IN solving Algebraical problems the chief difficulty of the beginner is to express the conditions of the question by means of symbols. A question proposed in algebraical symbols will frequently be found puzzling, when a similar arithmetical question would present no difficulty. Thus, the answer to the question "find a number greater than $x$ by $a$" may not be self-evident to the beginner, who would of course readily answer an analogous arithmetical question, "find a number greater than 50 by 6." The process of addition which gives the answer in the second case supplies the necessary hint; and, just as the number which is greater than 50 by 6 is $50+6$, so the number which is greater than $x$ by $a$ is $x+a$.

80. The following examples will perhaps be the best introduction to the subject of this chapter. After the first we leave to the student the choice of arithmetical instances, should he find them necessary.

*Example* 1. By how much does $x$ exceed 17?

Take a numerical instance; "by how much does 27 exceed 17?"

The answer obviously is 10, which is equal to $27-17$.

Hence the excess of $x$ over 17 is $x-17$.

Similarly the defect of $x$ from 17 is $17-x$.

*Example* 2. If $x$ is one *part* of 45 the other part is $45-x$.

*Example* 3. If $x$ is one *factor* of 45 the other factor is $\frac{45}{x}$.

*Example* 4. How far can a man walk in $a$ hours at the rate of 4 miles an hour?

In 1 hour he walks 4 miles,

In $a$ hours he walks $a$ times as far, that is, $4a$ miles.

*Example* 5. If £20 is divided equally among $y$ persons, the share of each is the total sum divided by the number of persons, or £$\frac{20}{y}$.

*Example* 6. If 17 be divided by 6 the quotient is 2, and the remainder 5,

that is,
$$\frac{17}{6}=2+\frac{5}{6}.$$

So if $N$ be divided by $D$, and the quotient be $Q$ and the remainder $R$, we have
$$\frac{N}{D}=Q+\frac{R}{D},$$
or
$$N=QD+R.$$

Thus, if the divisor is $x$, the quotient $y$, and the remainder $z$, the dividend is $xy+z$.

*Example* 7. $A$ and $B$ are playing for money; $A$ begins with £$p$ and $B$ with $q$ shillings: after $B$ has won £$x$, how many shillings has each?

What $B$ has won $A$ has lost,

$\therefore$ $A$ has $20(p-x)$ shillings,

$B$ has $q+20x$ shillings.

## EXAMPLES IX. a.

**1.** What must be added to $x$ to make $y$?

**2.** By what must 3 be multiplied to make $a$?

**3.** What dividend gives $b$ as the quotient when 5 is the divisor?

**4.** What is the defect of $2c$ from $3d$?

**5.** By how much does $3k$ exceed $k$?

**6.** If 100 be divided into two parts and one part be $x$ what is the other?

**7.** If $a$ be one factor of $b$, what is the other?

**8.** What number is less than 20 by $c$?

**9.** What is the price in pence of $a$ oranges at tenpence a dozen?

**10.** What is the price in pence of 100 oranges when $x$ cost sixpence?

**11.** If the difference of two numbers be 11, and if the smaller be $x$, what is the greater?

**12.** If the sum of two numbers be $c$ and one of them is 20, what is the other?

**13.** What is the excess of 90 over $x$?

**14.** By how much does $x$ exceed 30?

**15.** If 100 contains $x$ five times, what is the value of $x$?

**16.** What is the cost in pounds of 40 books at $x$ shillings each?

**17.** In $x$ years a man will be 36 years old, what is his present age?

**18.** How old will a man be in $a$ years if his present age is $x$ years?

**19.** If $x$ men take 5 days to reap a field, how long will one man take?

**20.** What value of $x$ will make $5x$ equal to 20?

**21.** What is the price in shillings of 120 apples, when the cost of a score is $x$ pence?

**22.** How many hours will it take to walk $x$ miles at 4 miles an hour?

**23.** How far can I walk in $x$ hours at the rate of $y$ miles an hour?

**24.** In $x$ days a man walks $y$ miles, what is his rate per day?

**25.** How many minutes will it take to walk $x$ miles at $a$ miles an hour?

**26.** A train goes $x$ miles an hour, how long does it take to go from Bristol to London, a distance of 120 miles?

**27.** How many miles is it between two places, if a train travelling $p$ miles an hour takes 5 hours to perform the journey?

**28.** What is the velocity in feet per second of a train which travels 30 miles in $x$ hours?

**29.** A man has $a$ crowns and $b$ florins, how many shillings has he?

**30.** If I spend $x$ shillings out of a sum of £20 how many shillings have I left?

**31.** Out of a purse containing £$a$ and $b$ shillings a man spends $c$ pence; express in pence the sum left.

**32.** By how much does $2x-5$ exceed $x+1$?

**33.** What number must be taken from $a-2b$ to leave $a-3b$?

**34.** If a bill is shared equally amongst $x$ persons and each pays 3*s.* 4*d.*, how many pence does the bill amount to?

**35.** If I give away $c$ shillings out of a purse containing $a$ sovereigns and $b$ florins, how many shillings have I left?

**36.** In how many weeks will $x$ horses eat 100 bushels of oats if one horse eats $y$ bushels a week?

**37.** If I spend $x$ shillings a week, how many pounds do I save out of a yearly income of £$y$?

**38.** A bookshelf contains $x$ Latin, $y$ Greek, and $z$ English books: if there are 100 books, how many are there in other languages?

**39.** I have $x$ pounds in my purse, $y$ shillings in one pocket, and $z$ pence in another: if I give away half-a-crown how many pence have I left?

**40.** In a class of $x$ boys, $y$ work at Classics, $z$ at Mathematics, and the rest are idle: what is the excess of workers over idlers?

81. We subjoin a few harder examples worked out in full.

*Example* 1. What is the present age of a man who $x$ years hence will be $m$ times as old as his son now aged $y$ years?

In $x$ years the son's age will be $y+x$ years; hence the father's age will be $m(y+x)$ years; therefore *now* the father's age is $m(y+x)-x$ years.

*Example* 2. Find the simple interest on £$k$ in $n$ years at $f$ per cent.

Interest on £100 for 1 year is £$f$,

∴ . . . . . . . . . £1 . . . . . . . . £$\frac{f}{100}$,

∴ . . . . . . . . . £$k$ . . . . . . . . £$\frac{kf}{100}$,

∴ Interest on £$k$ for $n$ years is £$\frac{nkf}{100}$.

*Example* 3. A room is $x$ yards long, $y$ feet broad, and $a$ feet high; find how many square yards of carpet will be required for the floor, and how many square yards of paper for the walls.

(1) The area of the floor is $3xy$ square feet;

∴ the number of square yards of carpet required is $\frac{3xy}{9}=\frac{xy}{3}$.

(2) The perimeter of the room is $2(3x+y)$ feet;

∴ the area of the walls is $2a(3x+y)$ square feet;

∴ number of square yards of paper required is $\frac{2a(3x+y)}{9}$.

*Example* 4. The digits of a number beginning from the left are $a$, $b$, $c$; what is the number?

Here $c$ is the digit in the units' place; $b$ standing in the tens' place represents $b$ tens; similarly $a$ represents $a$ hundreds.

The number is therefore equal to $a$ hundreds $+\, b$ tens $+\, c$ units

$$=100a+10b+c.$$

If the digits of the number are inverted, a new number is formed which is symbolically expressed by

$$100c+10b+a.$$

*Example* 5. What is (1) the sum, (2) the product of three consecutive numbers of which the least is $n$?

The numbers consecutive to $n$ are $n+1$, $n+2$;

$$\therefore \text{ the sum} = n+(n+1)+(n+2)$$
$$=3n+3.$$

And the product $=n(n+1)(n+2)$.

We may remark here that any *even* number may be denoted by $2n$, where $n$ is *any* positive whole number; for this expression is exactly divisible by 2.

Similarly, any odd number may be denoted by $2n+1$; for this expression when divided by 2 leaves remainder 1.

*Example* 6. How many days will $a$ men take to mow $b$ acres if $c$ boys can mow $a$ acres in $b$ days, and each man's work equals that of $n$ boys?

Since $c$ boys can mow $a$ acres in $b$ days;
$\therefore$ 1 boy . . . . . . . . . . . . . . . . . . . . $bc$ days,
$\therefore$ $n$ boys, or 1 man, . . . . . . . . . . . $\dfrac{bc}{n}$ days,
$\therefore$ $a$ men . . . . . . . . . . . . . . . . . . . . $\dfrac{bc}{an}$ days,
$\therefore$ $a$ men . . . . . . . . . . . . . 1 acre . . $\dfrac{bc}{a^2n}$ days;
therefore $a$ men can mow $b$ acres in $\dfrac{b^2c}{a^2n}$ days.

## EXAMPLES IX. b.

1. Write down four consecutive numbers of which $x$ is the least.

2. Write down three consecutive numbers of which $y$ is the greatest.

3. Write down five consecutive numbers of which $x$ is the middle one.

4. What is the next even number after $2n$?

5. What is the odd number next before $2x+1$?

6. Find the sum of three consecutive odd numbers of which the middle one is $2n+1$.

7. A man makes a journey of $x$ miles. He travels $a$ miles by coach, $b$ by train, and finishes the journey by boat. How far does the boat carry him?

8. A horse eats $a$ bushels and a donkey $b$ bushels of corn in a week; how many bushels will they together consume in $n$ weeks?

9. If a man was $x$ years old 5 years ago, how old will he be $y$ years hence?

10. A boy is $x$ years old, and five years hence his age will be half that of his father. How old is the father now?

11. What is the age of a man who $y$ years ago was $m$ times as old as a child then aged $x$ years?

12. $A$'s age is double $B$'s, $B$'s is three times $C$'s, and $C$ is $x$ years old: find $A$'s age.

13. What is the interest on £1000 in $b$ years at $c$ per cent.?

**14.** What is the interest on £$x$ in $a$ years at 5 per cent.?

**15.** What is the interest on £$50a$ in $a$ years at $a$ per cent.?

**16.** What is the interest on £$24xy$ in $x$ months at $y$ per cent. per annum?

**17.** A room is $x$ yards in length, and $y$ feet in breadth; how many square feet are there in the area of the floor?

**18.** A square room measures $x$ feet each way; how many square yards of carpet will be required to cover it?

**19.** A room is $p$ feet long and $x$ yards in width; how many yards of carpet two feet wide will be required for the floor?

**20.** What is the cost in pounds of carpeting a room $a$ yards long $b$ feet broad with carpet costing $c$ shillings a square yard?

**21.** How many yards of carpet $x$ inches wide will be required to cover the floor of a room $y$ feet long and $z$ feet broad?

**22.** A room is $a$ yards long and $b$ yards broad; in the middle there is a carpet $c$ feet square; how many square yards of oil-cloth will be required to cover the rest of the floor?

**23.** How many miles can a person walk in 45 minutes if he walks $a$ miles in $x$ hours?

**24.** How long will it take a person to walk $b$ miles if he walks 20 miles in $c$ hours?

**25.** If a train travels $a$ miles in $b$ hours, how many feet does it move through in one second?

**26.** A train is running with a velocity of $x$ feet per second; how many miles will it travel in $y$ hours?

**27.** How long will $x$ men take to mow $y$ acres of corn, if each man mows $z$ acres a day?

**28.** How many men will be required to do in $x$ hours what $y$ men do in $xz$ hours?

**29.** What is the rate per cent. which will produce £$y$ interest from a principal of £1000 in $r$ years?

**30.** Find in how many years a principal of £$a$ will produce £$p$ interest at $r$ per cent. per annum.

*82. In Example 6, Art. 80, we proved

$$\frac{N}{D}=Q+\frac{R}{D},$$

a result which gives in a single statement a general relation expressing the connection between a number, its divisor, and the resulting quotient and remainder.

This is an example of a very important class of Algebraical statements known as *formulae*, and it may be worth while briefly to foreshadow their use and application, not only in Algebra, but also in other branches of Mathematics and elementary Physics.

DEFINITION. A **formula** is a relation established by reasoning among certain quantities, any one of which may in turn be regarded as the unknown.

Thus in the formula above mentioned, if $Q$, $R$, and $D$ are given quantities, we have an equation to find the corresponding value of $N$. Or, a question may be proposed as follows: "By what must 96 be divided so as to give a quotient 5, and a remainder 11?" Here we have given $N=96$, $Q=5$, $R=11$, and therefore from the formula we obtain

$$\frac{96}{D}=5+\frac{11}{D},$$

whence $D=17$, the required divisor.

*83. A formula, it must be observed, includes all particular cases in one general statement; and so by the use of a single algebraical formula we are enabled briefly to express a whole class of results in a form at once simple, easily remembered, and easily applied. Experience will convince the student how much of the power and utility of Algebra lies in its formulae, and their ready application to all kinds of problems of every day occurrence.

It would be out of place here to make more than a passing allusion to other branches of Mathematics, or to Physical Science; but on account of the interest and importance of the subject, it may be useful to draw the reader's attention to a few of the more elementary formulae he is likely to meet with in his other studies.

(1) If a triangle on a base $b$, has a height $h$, its area ($A$) is given by the formula $A=\frac{1}{2}hb$.

(2) If a pyramid of height $h$ stands on a base whose area is $a^2$, its volume ($V$) is given by the formula

$$V=\tfrac{1}{3}a^2h.$$

In these cases any linear unit, inch, foot... being chosen, the superficial and solid units will be respectively the square and cubic inch, foot, ...; and in each of these formulae if two of the three quantities be given, the third is easily obtained by Arithmetic.

*Example.* The Great Pyramid of Egypt stands on a square base each side of which is 764 feet; and its height is 480 feet. Find the number of cubic feet of stone used in its construction.

$$\begin{aligned}\text{From the formula, } V&=\tfrac{1}{3}\times(764)^2\times 480\\ &=160\times 764\times 764\\ &=93391360 \text{ cubic feet.}\end{aligned}$$

*84. We have in this chapter given several examples involving space, velocity, and time; and all these can be solved without difficulty by common sense reasoning. At the same time we may remark that they are only particular cases of the general formula $s=vt$, in which $s$ denotes the space described by a body which moves with uniform velocity $v$ for a time $t$.

In this formula, if $t$ denotes the number of seconds the body has been in motion, and $v$ the number of feet passed over in one second, then $s$ is the space (in feet) described in $t$ seconds.

*Example*. If a train has a velocity of 75 feet a second, how long will it take to cross a viaduct which is 300 yards in length?

Substituting the values of $s$ and $v$ (expressed in feet) in the formula, we get

$$\begin{aligned} 900&=75t, \\ t&=\frac{900}{75} \\ &=12. \end{aligned}$$

Therefore the time is 12 seconds.

*85. Another very interesting case is that of a body falling vertically under the action of gravity.

It is proved in works on Dynamics that if a body fall freely from rest, and if $s$ denote the space (in feet) described in $t$ seconds,

$$s=\tfrac{1}{2}gt^2.$$

In this formula $g$ denotes the number of feet by which the velocity is increased in each successive second in consequence of the earth's attraction, and it is found by experiment that $g=32{\cdot}2$ nearly.

*Example* 1. A stone dropped from the Clifton suspension bridge takes 4 seconds before it reaches the water. Find the height of the bridge above the river.

From the above formula,

$$\begin{aligned} s&=\tfrac{1}{2}\times 32{\cdot}2\times(4)^2 \\ &=257{\cdot}6, \end{aligned}$$

and the height is therefore 257·6 feet.

*Example* 2. How long will it take a stone to reach the bottom of a well 144·9 feet deep?

From the formula,

$$\begin{aligned} 144{\cdot}9&=\tfrac{1}{2}\times 32{\cdot}2\times t^2; \\ \therefore\ t^2&=\frac{144{\cdot}9}{16{\cdot}1}=9; \\ \therefore\ t&=3. \end{aligned}$$

Therefore the time is 3 seconds.

# CHAPTER X.

## PROBLEMS LEADING TO SIMPLE EQUATIONS.

86. THE principles of the last Chapter may now be employed to solve various problems.

The method of procedure is as follows:

Represent the unknown quantity by a symbol $x$, and express in symbolical language the conditions of the question; we thus obtain a simple equation which can be solved by the methods already given in Chapter VIII.

*Example* 1. Find two numbers whose sum is 28, and whose difference is 4.

Let $x$ be the smaller number, then $x+4$ is the greater.

Their sum is $x+(x+4)$, which is to be equal to 28.

Hence
$$x+x+4=28;$$
$$2x=24;$$
$$\therefore\ x=12,$$
and
$$x+4=16,$$
so that the numbers are 12 and 16.

The beginner is advised to test his solution by proving that it satisfies the data of the question.

*Example* 2. Divide 60 into two parts, so that three times the greater may exceed 100 by as much as 8 times the less falls short of 200.

Let $x$ be the greater part, then $60-x$ is the less.

Three times the greater part is $3x$, and its excess over 100 is
$$3x-100.$$

Eight times the less is $8(60-x)$, and its defect from 200 is
$$200-8(60-x).$$

Whence the symbolical statement of the question is
$$3x-100=200-8(60-x);$$
$$3x-100=200-480+8x,$$
$$480-100-200=8x-3x,$$
$$5x=180;$$
$$\therefore\ x=36, \text{ the greater part,}$$
and
$$60-x=24, \text{ the less.}$$

*Example* 3. Divide £47 between $A$, $B$, $C$, so that $A$ may have £10 more than $B$, and $B$ £8 more than $C$.

Suppose that $C$ has $x$ pounds; then $B$ has $x+8$ pounds, and $A$ has $x+8+10$ pounds.

Hence
$$\begin{aligned} x+(x+8)+(x+8+10)&=47; \\ x+x+8+x+8+10&=47, \\ 3x&=21; \\ \therefore\quad x&=7; \end{aligned}$$
so that $C$ has £7, $B$ £15, $A$ £25.

*Example* 4. A person spent £28. 4*s*. in buying geese and ducks; if each goose cost 7*s*., and each duck 3*s*., and if the total number of birds bought was 108, how many of each did he buy?

In questions of this kind it is of essential importance to have all quantities expressed in the same denomination; in the present instance it will be convenient to express the money in shillings.

Let $x$ be the number of geese, then $108-x$ is the number of ducks.

Since each goose costs 7 shillings, $x$ geese cost $7x$ shillings.

And since each duck costs 3 shillings, $108-x$ ducks cost $3(108-x)$ shillings.

Therefore the amount spent is
$$7x+3(108-x) \text{ shillings},$$
but the question states that the amount is also £28. 4*s*., that is 564 shillings.

Hence
$$\begin{aligned} 7x+3(108-x)&=564; \\ 7x+324-3x&=564, \\ 4x&=240, \\ \therefore\quad x&=60, \text{ the number of geese,} \end{aligned}$$
and
$$108-x=48, \text{ the number of ducks.}$$

*Example* 5. $A$ is twice as old as $B$, ten years ago he was four times as old; what are their present ages?

Let $B$'s age be $x$ years, then $A$'s age is $2x$ years.

Ten years ago their ages were respectively, $x-10$ and $2x-10$ years; thus we have
$$\begin{aligned} 2x-10&=4(x-10); \\ 2x-10&=4x-40, \\ 2x&=30; \\ \therefore\quad x&=15, \end{aligned}$$
so that $B$ is 15 years old, $A$ 30 years.

NOTE. In the above examples the unknown quantity $x$ represents a *number* of pounds, ducks, years, &c.; and the student must be careful to avoid beginning a solution with a supposition of the kind, "let $x=A$'s share" or "let $x=$the ducks", or any statement so vague and inexact.

## EXAMPLES X. a.

**1.** One number exceeds another by 5, and their sum is 29; find them.

**2.** The difference between two numbers is 8; if 2 be added to the greater the result will be three times the smaller: find the numbers.

**3.** Find a number such that its excess over 50 may be greater by 11 than its defect from 89.

**4.** A man walks 10 miles, then travels a certain distance by train, and then twice as far by coach. If the whole journey is 70 miles, how far does he travel by train?

**5.** What two numbers are those whose sum is 58, and difference 28?

**6.** If 288 be added to a certain number, the result will be equal to three times the excess of the number over 12: find the number.

**7.** Twenty-three times a certain number is as much above 14 as 16 is above seven times the number: find it.

**8.** Divide 105 into two parts, one of which diminished by 20 shall be equal to the other diminished by 15.

**9.** Find three consecutive numbers whose sum shall equal 84.

**10.** The sum of two numbers is 8, and one of them with 22 added to it is five times the other: find the numbers.

**11.** Find two numbers differing by 10 whose sum is equal to twice their difference.

**12.** $A$ and $B$ begin to play each with £60. If they play till $A$'s money is double $B$'s, what does $A$ win?

**13.** Find a number such that if 5, 15, and 35 are added to it, the product of the first and third results may be equal to the square of the second.

**14.** The difference between the squares of two consecutive numbers is 121: find the numbers.

**15.** The difference of two numbers is 3, and the difference of their squares is 27: find the numbers.

**16.** Divide £380 between $A$, $B$, and $C$, so that $B$ may have £30 more than $A$, and $C$ may have £20 more than $B$.

**17.** A sum of £8. 17*s*. is made up of 124 coins which are either florins or shillings: how many are there of each?

**18.** If silk costs six times as much as linen, and I spend £9. 8*s*. in buying 23 yards of silk and 50 yards of linen; find the cost of each per yard.

**19.** A father is four times as old as his son: in 24 years he will only be twice as old; find their ages.

**20.** $A$ is 25 years older than $B$, and $A$'s age is as much above 20 as $B$'s is below 85: find their ages.

**21.** $A$'s age is six times $B$'s, and fifteen years hence $A$ will be three times as old as $B$: find their ages.

**22.** A sum of £4. 5*s.* was paid in crowns, half-crowns, and shillings. The number of half-crowns used was four times the number of crowns and twice the number of shillings; how many were there of each?

**23.** The sum of the ages of $A$ and $B$ is 30 years, and five years hence $A$ will be three times as old as $B$: find their present ages.

**24.** In a cricket match the byes were double of the wides, and the remainder of the score was greater by three than twelve times the number of byes. If the whole score was 138, how were the runs obtained?

**25.** The length of a room exceeds its breadth by 3 feet; if the length had been increased by 3 feet, and the breadth diminished by 2 feet, the area would not have been altered: find the dimensions.

**26.** The length of a room exceeds its breadth by 8 feet; if each had been increased by 2 feet, the area would have been increased by 60 square feet: find the original dimensions of the room.

87. We add some problems which lead to equations with fractional coefficients.

*Example* 1. Find two numbers which differ by 4, and such that one-half of the greater exceeds one-sixth of the less by 8.

Let $x$ be the smaller number, then $x+4$ is the greater.

One-half of the greater is represented by $\frac{1}{2}(x+4)$, and one-sixth of the less by $\frac{1}{6}x$.

Hence $$\frac{1}{2}(x+4)-\frac{1}{6}x=8;$$

multiplying by 6, $$3x+12-x=48;$$

$$\therefore\ 2x=36;$$

$$\therefore\ x=18, \text{ the less number,}$$

and $$x+4=22, \text{ the greater.}$$

*Example* 2. $A$ has £9, and $B$ has 4 guineas; after $B$ has won from $A$ a certain sum, $A$ has then five-sixths of what $B$ has; how much did $B$ win?

Suppose that $B$ wins $x$ *shillings*, $A$ has then $180-x$ shillings, and $B$ has $84+x$ shillings;

Hence $$180-x=\frac{5}{6}(84+x);$$

$$1080-6x=420+5x,$$

$$11x=660;$$

$$\therefore\ x=60.$$

Therefore $B$ wins 60 shillings, or £3.

## EXAMPLES X. b.

**1.** Find a number such that the sum of its sixth and ninth parts may be equal to 15.

**2.** What is the number whose eighth, sixth, and fourth parts together make up 13?

**3.** There is a number whose fifth part is less than its fourth part by 3: find it.

**4.** Find a number such that six-sevenths of it shall exceed four-fifths of it by 2.

**5.** The fifth, fifteenth, and twenty-fifth parts of a number together make up 23: find the number.

**6.** Two consecutive numbers are such that one-fourth of the less exceeds one-fifth of the greater by 1: find the numbers.

**7.** Two numbers differ by 28, and one is eight-ninths of the other; find them.

**8.** There are two consecutive numbers such that one-fifth of the greater exceeds one-seventh of the less by 3: find them.

**9.** Find three consecutive numbers such that if they be divided by 10, 17, and 26 respectively, the sum of the quotients will be 10.

**10.** $A$ and $B$ begin to play with equal sums, and when $B$ has lost five-elevenths of what he had to begin with, $A$ has gained £6 more than half of what $B$ has left: what had they at first?

**11.** From a certain number 3 is taken, and the remainder is divided by 4; the quotient is then increased by 4 and divided by 5 and the result is 2: find the number.

**12.** In a cellar one-fifth of the wine is port and one-third claret: besides this it contains 15 dozen of sherry and 30 bottles of spirits. How much port and claret does it contain?

**13.** Two-fifths of $A$'s money is equal to $B$'s, and seven-ninths of $B$'s is equal to $C$'s: in all they have £770, what have they each?

**14.** $A$, $B$, and $C$ have £1285 between them: $A$'s share is greater than five-sixths of $B$'s by £25, and $C$'s is four-fifteenths of $B$'s: find the share of each.

**15.** A man sold a horse for £35 and half as much as he gave for it, and gained thereby ten guineas: what did he pay for the horse?

**16.** The width of a room is two-thirds of its length. If the width had been 3 feet more, and the length 3 feet less, the room would have been square; find its dimensions.

**17.** What is the property of a person whose income is £430, when he has two-thirds of it invested at 4 per cent., one-fourth at 3 per cent., and the remainder at 2 per cent.?

**18.** I bought a certain number of apples at three a penny, and five-sixths of that number at four a penny; by selling them at sixteen for sixpence I gained $3\frac{1}{2}d.$; how many apples did I buy?

# CHAPTER XI.

## HIGHEST COMMON FACTOR. LOWEST COMMON MULTIPLE.

### SIMPLE EXPRESSIONS.

88. DEFINITION. The **highest common factor** of two or more algebraical expressions is the expression of highest dimensions [Art. 10] which divides each of them without remainder.

The abbreviation H.C.F. is sometimes used instead of the words *highest common factor.*

89. In the case of *simple expressions* the highest common factor can be written down by inspection.

*Example* 1. The highest common factor of $a^4$, $a^3$, $a^2$, $a^6$ is $a^2$.

*Example* 2. The highest common factor of $a^3b^4$, $ab^5c^2$, $a^2b^7c$ is $ab^4$; for $a$ is the highest power of $a$ that will divide $a^3$, $a$, $a^2$; $b^4$ is the highest power of $b$ that will divide $b^4$, $b^5$, $b^7$; and $c$ is not a *common* factor.

90. If the expressions have numerical coefficients, find by Arithmetic their greatest common measure, and prefix it as a coefficient to the algebraical highest common factor.

*Example.* The highest common factor of $21a^4x^3y$, $35a^2x^4y$, $28a^3xy^4$ is $7a^2xy$; for it consists of the product of

(1) the numerical greatest common measure of the coefficients;

(2) the highest power of each letter which divides every one of the given expressions.

## EXAMPLES XI. a.

Find the highest common factor of

**1.** $4ab^2$, $2a^2b$. **2.** $3x^2y^2$, $x^3y^2$. **3.** $6xy^2z$, $8x^2y^3z^2$.

**4.** $abc$, $2ab^2c$. **5.** $5a^3b^3$, $15abc^2$. **6.** $9x^2y^2z^2$, $12xy^3z$.

**7.** $4a^2b^3c^2$, $6a^3b^2c^3$. **8.** $7a^2b^4c^5$, $14ab^2c^3$.

**9.** $15x^4y^3z^2$, $12x^2yz^2$. **10.** $8a^2x$, $6abxy$, $10abx^3y^2$.

**11.** $49ax^2$, $63ay^2$, $56az^2$. **12.** $17ab^2c$, $34a^2bc$, $51abc^2$.

**13.** $a^3x^2y^2$, $b^3xy^2$, $c^3x^2y$.

**14.** $24a^2b^3c^3$, $64a^3b^3c^2$, $48a^3b^2c^3$.

**15.** $25xy^2z$, $100x^2yz$, $125xy$.

**16.** $a^2bpxy$, $b^2qxy$, $a^3bxr^2$.

**17.** $15a^5b^3c^7$, $60a^3b^7c^8$, $25a^4b^5c^2$.

**18.** $35a^2c^3b$, $42a^3cb^2$, $30ac^2b^3$.

**19.** $24a^3b^2c^3$, $16a^3b^4c^2$, $40a^2b^3c^5$.

**20.** $66a^4b^2c^3$, $44a^3b^4c^2$, $24a^2b^3c^4$.

## LOWEST COMMON MULTIPLE.

91. DEFINITION. The **lowest common multiple** of two or more algebraical expressions is the expression of lowest dimensions which is divisible by each of them without remainder.

The abbreviation L.C.M. is sometimes used instead of the words *lowest common multiple.*

92. In the case of *simple expressions* the lowest common multiple can be written down by inspection.

*Example* 1. The lowest common multiple of $a^4$, $a^3$, $a^2$, $a^6$ is $a^6$.

*Example* 2. The lowest common multiple of $a^3b^4$, $ab^5$, $a^2b^7$ is $a^3b^7$; for $a^3$ is the lowest power of $a$ that is divisible by each of the quantities $a^3$, $a$, $a^2$; and $b^7$ is the lowest power of $b$ that is divisible by each of the quantities $b^4$, $b^5$, $b^7$.

93. If the expressions have numerical coefficients, find by Arithmetic their least common multiple, and prefix it as a coefficient to the algebraical lowest common multiple.

*Example.* The lowest common multiple of $21a^4x^3y$, $35a^2x^4y$, $28a^3xy^4$ is $420a^4x^4y^4$; for it consists of the product of

(1) the numerical least common multiple of the coefficients;

(2) the lowest power of each letter which is divisible by every power of that letter occurring in the given expressions.

## EXAMPLES XI. b.

Find the lowest common multiple of

**1.** $abc$, $2a^2$.

**2.** $x^3y^2$, $xyz$.

**3.** $3x^2yz$, $4x^3y^3$.

**4.** $5a^2bc^3$, $4ab^2c$.

**5.** $3a^4b^2c^3$, $5a^2b^3c^5$.

**6.** $12ab$, $8xy$.

**7.** $ac$, $bc$, $ab$.

**8.** $a^2c$, $bc^2$, $cb^2$.

**9.** $2ab$, $3bc$, $4ca$.

**10.** $2x$, $3y$, $4z$.

**11.** $3x^2$, $4y^2$, $3z^2$.

**12.** $7a^2$, $2ab$, $3b^3$.

**13.** $a^2bc$, $b^2ca$, $c^2ab$.

**14.** $5a^2c$, $6cb^2$, $3bc^2$.

**15.** $2x^2y^3$, $3xy$, $4x^3y^4$.

**16.** $7x^4y$, $8xy^5$, $2x^3y^3$.

**17.** $35a^2c^3b$, $42a^3cb^2$, $30ac^2b^3$.

**18.** $66a^4b^2c^3$, $44a^3b^4c^2$, $24a^2b^3c^4$.

**19.** $7a^2b$, $4ac^2$, $6ac^3$, $21bc$.

**20.** $8a^2b^2$, $24a^4b^2c^2$, $18abc^3$.

## EXAMPLES XI. c.

Find both the highest common factor and the lowest common multiple of

1. $2abc$, $3ca$, $4bca$.
2. $2xy$, $4yz$, $6zxy$.
3. $9abc$, $3b^2c$, $cab$.
4. $13a^2bc$, $39a^3bc^2$.
5. $17xyz^2$, $51x^2y$.
6. $15x^3y^3z$, $25xy^3z^2$.
7. $3ab$, $2bc$, $5cab$.
8. $17m^2n^4p^2$, $51m^4p^4$.
9. $x^3y^2$, $y^2z^4$, $z^4x^3y^5$.
10. $5mnp$, $4npq$, $3mpq$.
11. $29mnp$, $87npq$.
12. $39a^2b^2$, $52c^2a^2$.
13. $21a^2b^3c$, $28xyz^2$.
14. $35axy$, $45xyz$.
15. $27m^2n^2p^2$, $81n^2x^2$.
16. $32a^4b^3c$, $48abc^5$, $16a^4c^3$.
17. $57a^2x^3y$, $76b^3x^2$.
18. $49bc^3$, $21a^2b^2$, $56ca^3$.
19. $15p^3q^4$, $20m^2p^2q^3$, $30mp^3$.
20. $72k^2m^3n^4$, $108k^3m^2n^5$.

# CHAPTER XII.

## ELEMENTARY FRACTIONS.

### SIMPLE EXPRESSIONS.

94. IN this Chapter we propose to deal only with the easier kinds of fractions, where the numerator and denominator are simple expressions.

Their reduction and simplification will be performed by the usual arithmetical rules. The proofs of these rules we reserve for a later chapter where the subject of fractions will be treated more fully.

95. **Rule.** *To reduce a fraction to its lowest terms: divide numerator and denominator by every factor which is common to them both, that is by their highest common factor.*

Dividing numerator and denominator of a fraction by a common factor is called *cancelling* that factor.

*Examples.* (1) $\frac{6a^2c}{9ac^2}=\frac{2a}{3c}$.

(2) $\frac{7x^2yz}{28x^3yz^2}=\frac{1}{4xz}$.

(3) $\frac{35a^5b^3c}{7ab^2c}=\frac{5a^4b}{1}=5a^4b$.

## EXAMPLES XII. a.

Reduce to lowest terms

1. $\frac{3a}{6ab}$. 2. $\frac{4a^2}{16ab}$. 3. $\frac{2xy^2}{5x^2y}$. 4. $\frac{3abc}{15a^2b^2c}$.

5. $\frac{x^2yz^3}{x^3y^2z}$. 6. $\frac{15ab}{25bc}$. 7. $\frac{21x^2y^2}{28y^2z^2}$. 8. $\frac{8a^2b}{12b^2c}$.

9. $\frac{12mn^2p}{15m^2np^2}$. 10. $\frac{15m^2p^3}{18n^4p}$. 11. $\frac{abc^2}{a^3b^2c}$.

12. $\frac{3x^2yz^3}{5xy^4z^2}$. 13. $\frac{2xy^3z^4}{4x^2y^2z}$. 14. $\frac{5a^3b^2c^4}{15ab^4c}$.

15. $\frac{mn^4pq}{m^2n^3p^4}$. 16. $\frac{4m^3n^2p^5}{6m^4np^2}$. 17. $\frac{15ax^3y^2}{25a^2xy^6}$.

18. $\frac{39a^2b^4c^3}{52a^3b^5c^4}$. 19. $\frac{38k^2p^3m^4}{57k^3pm^2}$. 20. $\frac{46x^3y^4z^5}{69x^2y^3z^4}$.

### Multiplication and Division of Fractions.

96. **Rule.** *To multiply algebraical fractions: as in Arithmetic, multiply together all the numerators for a new numerator, and all the denominators for a new denominator.*

*Example* 1. $\frac{2a}{3b}\times\frac{5x^2}{2a^2b}\times\frac{3b^2}{2x}=\frac{2a\times5x^2\times3b^2}{3b\times2a^2b\times2x}=\frac{5x}{2a}$,

by cancelling like factors in numerator and denominator.

*Example* 2. $\frac{3a^2b}{5c^2}\times\frac{7bc}{3a^3}\times\frac{5ca}{7b^2}=1$,

all the factors cancelling each other.

97. RULE. *To divide one fraction by another: invert the divisor and proceed as in multiplication.*

*Example.*

$$\frac{7a^3}{4x^3y^2} \times \frac{6c^3x}{5ab^2} \div \frac{28a^2c^2}{15b^2xy^2}$$

$$= \frac{7a^3}{4x^3y^2} \times \frac{6c^3x}{5ab^2} \times \frac{15b^2xy^2}{28a^2c^2}$$

$$= \frac{9c}{8x},$$

all the other factors cancelling each other.

## EXAMPLES XII. b.

Simplify the following expressions :

1. $\dfrac{2ab}{3cd} \times \dfrac{c^2d^3}{ab^2}$.

2. $\dfrac{12a^2bc}{8ab^3} \times \dfrac{24ab^2}{36bc^2}$.

3. $\dfrac{15xyz^3}{a^2bc} \times \dfrac{3a^3x}{5yz}$.

4. $\dfrac{7a^2b^3}{9ax^2y} \times \dfrac{18x^2c}{15ac^4}$.

5. $\dfrac{8m^2n^3}{5x^2yz} \times \dfrac{15xyz^3}{16mn^2}$.

6. $\dfrac{21k^2p^3}{13mn^2} \times \dfrac{39n^3m^2}{28p^2k^3}$.

7. $\dfrac{3a^2b}{4b^3c} \times \dfrac{2c^2}{8a^3} \div \dfrac{6ac}{16b^2x}$.

8. $\dfrac{2x^2y}{3yz} \times \dfrac{5z^2x}{7xy^2} \div \dfrac{21x^2y^3z^2}{40xy^2z}$.

9. $\dfrac{7m^2p}{17x^2y} \times \dfrac{51y^3z}{21p^2n} \div \dfrac{m^2x^2}{pyz}$.

10. $\dfrac{26xk^2p^3}{58mp^4} \times \dfrac{2xk^3}{13pkm} \div \dfrac{2x^2k^4}{87m^2p^2}$.

11. $\dfrac{15b^2}{40c} \times \dfrac{27c^2}{81d^3} \div \dfrac{abc}{14d^3}$.

12. $\dfrac{b^2}{3c} \times \dfrac{4c^2}{5d^3} \div \dfrac{16a^2b^2c^2}{15d^5}$.

13. $\dfrac{8ax^2}{7by} \times \dfrac{49cy^2}{64dx^3}$.

14. $\dfrac{15abc}{16xyz} \times \dfrac{128x^3y^2z^2}{100a^2bc}$.

15. $\dfrac{45a^2b^3c^4}{27x^4y^3z} \times \dfrac{243xy^2z^3}{180a^2bc^3}$.

16. $\dfrac{104xyzk^2p}{28xy^2kp^2} \times \dfrac{56y^3z^5p}{26y^2z^6k}$.

17. $\dfrac{m^2}{8n} \times \dfrac{36p^3q^2}{81mn} \div \dfrac{15mpx^5}{27n^2x^3y}$.

18. $\dfrac{a^3}{b^3} \times \dfrac{xy^2}{ab} \times \dfrac{pb^2}{ax} \div \dfrac{ap}{b^2}$.

### REDUCTION TO A COMMON DENOMINATOR.

98. In order to find the sum or difference of any fractions, we must, as in Arithmetic, first reduce them to a common denominator; and it is most convenient to take the lowest common multiple of the denominators of the given fractions.

*Example.* Express with lowest common denominator the fractions

$$\frac{a}{3xy},\ \frac{b}{6xyz},\ \frac{c}{2yz}.$$

The lowest common multiple of the denominators is $6xyz$. Multiplying the numerator of each fraction by the factor which is required to make its denominator $6xyz$, we have the equivalent fractions

$$\frac{2az}{6xyz},\ \frac{b}{6xyz},\ \frac{3cx}{6xyz}.$$

NOTE. The same result would clearly be obtained by dividing the lowest common denominator by each of the denominators in turn, and multiplying the corresponding numerators by the respective quotients.

## EXAMPLES XII. c.

Express as equivalent fractions with common denominator:

1. $\frac{2x}{a},\ \frac{y}{2a}.$
2. $\frac{4x}{3y},\ \frac{y}{x^2}.$
3. $\frac{a}{2b},\ \frac{b}{c}.$
4. $\frac{a}{b},\ \frac{c}{d},\ 2.$
5. $\frac{2a}{b},\ \frac{b}{3c}.$
6. $\frac{m}{4n},\ \frac{p}{5n}.$
7. $\frac{k}{2x},\ \frac{p}{3x}.$
8. $\frac{m}{3x},\ \frac{n}{6x}.$
9. $\frac{a}{bc},\ \frac{b}{ca}.$
10. $\frac{a}{x},\ \frac{b}{x^2}.$
11. $\frac{2}{x},\ \frac{3}{y}.$
12. $\frac{x}{y},\ \frac{y}{x},\ 3x.$
13. $\frac{2x}{3y},\ \frac{3y}{2x}.$
14. $\frac{4a}{5b},\ \frac{3a}{10c}.$
15. $\frac{3a}{7b},\ \frac{5b}{21c}.$

### ADDITION AND SUBTRACTION OF FRACTIONS.

99. RULE. *To add or subtract fractions: express all the fractions with their lowest common denominator; form the sum or difference of the numerators, and retain the common denominator.*

*Example* 1. Simplify $\frac{5x}{8}+\frac{3}{4}x-\frac{7x}{6}.$

The least common denominator is 12.

The expression $=\frac{20x+9x-14x}{12}=\frac{15x}{12}=\frac{5x}{4}.$

*Example* 2. Simplify $\frac{3ab}{5x}-\frac{ab}{2x}-\frac{1}{10}\cdot\frac{ab}{x}.$

The expression $=\frac{6ab-5ab-ab}{10x}=\frac{0}{10x}=0.$

*Example* 3. Simplify $\dfrac{2x}{a^2c^2} - \dfrac{y}{3ca^3}$.

The expression $= \dfrac{6ax - cy}{3a^3c^2}$, and admits of no further simplification.

**Note.** The beginner must be careful to distinguish between **erasing equal terms with different signs**, as in Example 2, and **cancelling equal factors** in the course of multiplication, or in reducing fractions to lowest terms. Moreover, in simplifying fractions he must remember that a factor can only be removed from numerator and denominator when it divides each *taken as a whole*.

Thus in $\dfrac{6ax - cy}{3a^3c^2}$, $c$ cannot be cancelled because it only divides $cy$ and not the *whole* numerator. Similarly $a$ cannot be cancelled because it only divides $6ax$ and not the whole numerator. The fraction is therefore in its simplest form.

When no denominator is expressed the denominator 1 may be understood.

*Example.* $3x - \dfrac{a^2}{4y} = \dfrac{3x}{1} - \dfrac{a^2}{4y} = \dfrac{12xy - a^2}{4y}$.

## EXAMPLES XII. d.

Simplify the following expressions:

1. $\dfrac{x}{2} + \dfrac{x}{3}$. 2. $\dfrac{y}{4} - \dfrac{y}{5}$. 3. $\dfrac{a}{3} - \dfrac{a}{4}$. 4. $\dfrac{2x}{3} - \dfrac{5}{x}$.

5. $\dfrac{x}{2} + \dfrac{y}{5}$. 6. $\dfrac{a}{4} - \dfrac{b}{6}$. 7. $\dfrac{m}{8} - \dfrac{n}{12}$. 8. $\dfrac{2m}{15} - \dfrac{n}{5}$.

9. $\dfrac{x}{7} - \dfrac{y}{21}$. 10. $\dfrac{a}{13} + \dfrac{b}{39}$. 11. $\dfrac{p}{16} - \dfrac{q}{48}$. 12. $\dfrac{5m}{12} - \dfrac{n}{36}$.

13. $\dfrac{2x}{3} + \dfrac{4x}{5}$. 14. $\dfrac{5x}{4} - \dfrac{4x}{5}$. 15. $\dfrac{5x}{6} - \dfrac{7x}{12}$. 16. $\dfrac{2a}{5} - \dfrac{4b}{15}$.

17. $\dfrac{a}{2} - \dfrac{a}{3} + \dfrac{a}{5}$. 18. $\dfrac{x}{4} - \dfrac{x}{8} + \dfrac{x}{12}$. 19. $\dfrac{x}{3} + \dfrac{x}{6} - \dfrac{x}{9}$.

20. $\dfrac{2x}{3} - \dfrac{x}{6} + \dfrac{3x}{4}$. 21. $\dfrac{5x}{6} - \dfrac{x}{12} + \dfrac{x}{9}$. 22. $\dfrac{7x}{8} + \dfrac{x}{12} - \dfrac{x}{4}$.

23. $\dfrac{x}{a} - \dfrac{y}{b}$. 24. $\dfrac{3x}{a} + \dfrac{2y}{3b}$. 25. $a + \dfrac{b}{c}$. 26. $x - \dfrac{y}{z}$.

27. $\dfrac{a}{3} - \dfrac{b^2}{a}$. 28. $a^2 + \dfrac{b^3}{a}$. 29. $\dfrac{x}{2} - \dfrac{y^3}{x^2}$. 30. $p^3 - \dfrac{k^5}{p^2}$.

# CHAPTER XIII.

## SIMULTANEOUS EQUATIONS.

100. CONSIDER the equation $2x+5y=23$, which contains *two* unknown quantities.

From this we get $5y=23-2x$,

that is, $y=\dfrac{23-2x}{5}$ ..........................(1).

From this it appears that for every value we choose to give to $x$ there will be one corresponding value of $y$. Thus we shall be able to find as many pairs of values as we please which satisfy the given equation.

For instance, if $x=1$, then from (1) $y=\dfrac{21}{5}$.

Again, if $x=-2$, then $y=\dfrac{27}{5}$; and so on.

But if also we have a second equation of the same kind, such as

$$3x+4y=24,$$

we have from this $y=\dfrac{24-3x}{4}$ ..........................(2).

If now we seek values of $x$ and $y$ which satisfy *both* equations, the values of $y$ in (1) and (2) must be identical.

Therefore $\dfrac{23-2x}{5}=\dfrac{24-3x}{4}$.

Multiplying up, $92-8x=120-15x$;

$$\therefore \quad 7x=28;$$

$$\therefore \quad x=4.$$

Substituting this value in the first equation, we have

$$8+5y=23;$$

$$\therefore \quad 5y=15;$$

$$\therefore \quad \left.\begin{aligned} y&=3,\\ \text{and} \quad x&=4. \end{aligned}\right\}$$

Thus, if both equations are to be satisfied by the *same* values of $x$ and $y$, there is only one solution possible.

101. DEFINITION. When two or more equations are satisfied by the same values of the unknown quantities they are called **simultaneous equations.**

We proceed to explain the different methods for solving simultaneous equations. In the present chapter we shall confine our attention to the simpler cases in which the unknown quantities are involved in the first degree.

102. In the example already worked we have used the method of solution which best illustrates the meaning of the term *simultaneous equation;* but in practice it will be found that this is rarely the readiest mode of solution. It must be borne in mind that since the two equations are simultaneously true, *any* equation formed by combining them will be satisfied by the values of $x$ and $y$ which satisfy the original equations. Our object will always be to obtain an equation which involves *one only* of the unknown quantities.

103. The process by which we get rid of either of the unknown quantities is called **elimination**, and it must be effected in different ways according to the nature of the equations proposed.

*Example* 1. Solve $3x+7y=27$ .............................. (1),

$5x+2y=16$ .............................. (2).

To eliminate $x$ we multiply (1) by 5 and (2) by 3, so as to make the coefficients of $x$ in both equations equal. This gives

$$15x+35y=135,$$

$$15x+\ \ 6y=48;$$

*subtracting,*

$$29y=87;$$

$$\therefore\ y=3.$$

To find $x$, substitute this value of $y$ in *either* of the given equations;

Thus from (1) $3x+21=27;$

$$\left.\begin{aligned}\therefore\ x&=2,\\ y&=3.\end{aligned}\right\}$$

and

NOTE. When one of the unknowns has been found, it is immaterial which of the equations we use to complete the solution. Thus, in the present example, if we substitute 3 for $y$ in (2), we have

$$5x+6=16;$$

$$\therefore\ x=2, \text{ as before.}$$

*Example* 2. Solve $7x+2y=47$ ............................(1),

$5x-4y=1$ ............................(2).

Here it will be more convenient to eliminate $y$.

Multiplying (1) by 2, $14x+4y=94$,

and from (2) $5x-4y=1$;

*adding*, $19x=95$;

$\therefore\ x=5$.

Substitute this value in (1),

$\therefore\ 35+2y=47$;

$\therefore\ y=6,$

and $x=5.$

NOTE. *Add* when the coefficients of one unknown are equal and *unlike* in sign; *subtract* when the coefficients are equal and *like* in sign.

*Example* 3. Solve $2x=5y+1$ ............................(1),

$24-7x=3y$ ............................(2).

Here we can eliminate $x$ by substituting in (2) its value obtained from (1). Thus

$$24-\frac{7}{2}(5y+1)=3y;$$

$$\therefore\ 48-35y-7=6y;$$

$$\therefore\ 41=41y;$$

$\therefore\ y=1,$

and from (1) $x=3.$

104. Any one of the methods given above will be found sufficient; but there are certain arithmetical artifices which will frequently shorten the work. The most useful of these will be shewn in the following examples.

*Example* 1. Solve $171x-213y=642$ ............................(1),

$114x-326y=244$ ............................(2).

Noticing that 171 and 114 contain a common factor 57, we shall make the coefficients of $x$ in the two equations equal to the *least common multiple* of 171 and 114 if we multiply (1) by 2 and (2) by 3.

Thus $342x-426y=1284$,

$342x-978y=732$;

subtracting, $552y=552$;

that is, $y=1,$

whence $x=5.$

*Example* 2. Solve $127x+59y=1928 \quad\text{..........}\quad (1),$

$59x+127y=1792 \quad\text{..........}\quad (2).$

By addition, $186x+186y=3720;$

$\therefore\ x+y=20 \quad\text{..........}\quad (3).$

Subtracting (2) from (1), $68x-68y=136;$

$\therefore\ x-y=2 \quad\text{..........}\quad (4).$

Thus, by an easy combination of (1) and (2), the problem is reduced to the solution of the equations (3) and (4). From these we obtain by addition $2x=22$, and by subtraction $2y=18$.

Therefore $x=11$, and $y=9$.

## EXAMPLES XIII. a.

Solve the equations:

1. $3x+4y=10,\quad 4x+y=9.$
2. $x+2y=13,\quad 3x+y=14.$
3. $4x+7y=29,\quad x+3y=11.$
4. $2x-y=9,\quad 3x-7y=19.$
5. $5x+6y=17,\quad 6x+5y=16.$
6. $2x+y=10,\quad 7x+8y=53.$
7. $8x-y=34,\quad x+8y=53.$
8. $15x+7y=29,\quad 9x+15y=39.$
9. $14x-3y=39,\quad 6x+17y=35.$
10. $28x-23y=33,\quad 63x-25y=101.$
11. $35x+17y=86,\quad 56x-13y=17.$
12. $15x+77y=92,\quad 55x-33y=22.$
13. $5x-7y=0,\quad 7x+5y=74.$
14. $21x-50y=60,\quad 28x-27y=199.$
15. $39x-8y=99,\quad 52x-15y=80.$
16. $5x=7y-21,\quad 21x-9y=75.$
17. $6y-5x=18,\quad 12x-9y=0.$
18. $8x=5y,\quad 13x=8y+1.$
19. $3x=7y,\quad 12y=5x-1.$
20. $19x+17y=0,\quad 2x-y=53.$
21. $93x+15y=123,\quad 15x+93y=201.$

105. We add a few cases in which, before proceeding to solve, it will be necessary to simplify the equations.

*Example* 1. Solve $5(x+2y)-(3x+11y)=14 \quad\text{..........}\quad (1),$

$7x-9y-3(x-4y)=38 \quad\text{..........}\quad (2).$

From (1) $5x+10y-3x-11y=14;$

$\therefore\ 2x-y=14 \quad\text{..........}\quad (3).$

From (2) $7x-9y-3x+12y=38;$

$\therefore\ 4x+3y=38 \quad\text{..........}\quad (4).$

From (3) $6x-3y=42$

and hence we may find $x=8$, and $y=2$.

*Example* 2. Solve $3x-\frac{y-5}{7}=\frac{4x-3}{2}$ ..........................(1),

$\frac{3y+4}{5}-\frac{1}{3}(2x-5)=y$ ......................... (2).

Clear of fractions. Thus

from (1) $42x-2y+10=28x-21;$

$\therefore\ 14x-2y=-31$ ..............................(3).

From (2) $9y+12-10x+25=15y;$

$\therefore\ 10x+6y=37$ .............................. (4).

Eliminating $y$ from (3) and (4), we find that

$$x=-\frac{14}{13}.$$

Eliminating $x$ from (3) and (4), we find that

$$y=\frac{207}{26}.$$

NOTE. Sometimes, as in the present instance, the value of the second unknown is more easily found by elimination than by substituting the value of the unknown already found.

## EXAMPLES XIII. b.

1. $\frac{2x}{3}+y=16,$ $\quad x+\frac{y}{4}=14.$

2. $\frac{x}{5}+\frac{y}{2}=5,$ $\quad x-y=4.$

3. $\frac{5x}{6}-y=3,$ $\quad x-\frac{5y}{6}=8.$

4. $x-y=5,$ $\quad \frac{x}{4}-\frac{y}{5}=2.$

5. $\frac{x}{9}+\frac{y}{7}=10,$ $\quad \frac{x}{3}+y=50.$

6. $x=3y,$ $\quad \frac{x}{3}+y=34.$

7. $\frac{2}{5}x-\frac{1}{12}y=3,$ $\quad 4x-y=20.$

8. $\frac{1}{2}x-\frac{1}{5}y=4,$ $\quad \frac{1}{7}x+\frac{1}{15}y=3.$

9. $2x+y=0,$ $\quad \frac{1}{2}y-3x=8.$

10. $\frac{x}{7}+\frac{y}{5}=1\frac{3}{7},$ $\quad x+\frac{y}{3}=4\frac{2}{3}.$

11. $3x-7y=0,$ $\quad \frac{2}{7}x+\frac{5}{3}y=7.$

12. $\frac{x}{5}-\frac{y}{4}=0,$ $\quad 3x+\frac{1}{2}y=17.$

13. $\frac{x}{3}+\frac{y}{4}=3x-7y-37=0.$ 14. $\frac{x+1}{10}=\frac{3y-5}{2}=\frac{x-y}{8}.$

15. $\frac{x+3}{5}=\frac{8-y}{4}=\frac{3(x+y)}{8}.$ 16. $\frac{x}{13}-\frac{y}{7}=6x-10y-8=0.$

106. In order to solve simultaneous equations which contain two unknown quantities we have seen that we must have two equations. Similarly we find that in order to solve simultaneous equations which contain three unknown quantities we must have three equations.

**Rule.** *Eliminate one of the unknowns from any pair of the equations, and then eliminate the same unknown from another pair. Two equations involving two unknowns are thus obtained, which may be solved by the rules already given. The remaining unknown is then found by substituting in any one of the given equations.*

*Example* 1. Solve $6x+2y-5z=13$ .......................... (1),

$3x+3y-2z=13$ .......................... (2),

$7x+5y-3z=26$ .......................... (3).

Choose $y$ as the unknown to be eliminated.

Multiply (1) by 3 and (2) by 2,

$$18x+6y-15z=39,$$
$$6x+6y-4z=26;$$

subtracting, $12x-11z=13$ .......................... (4).

Again, multiply (1) by 5 and (3) by 2,

$$30x+10y-25z=65,$$
$$14x+10y-6z=52;$$

subtracting, $16x-19z=13$ .......................... (5).

Multiply (4) by 4 and (5) by 3,

$$48x-44z=52,$$
$$48x-57z=39;$$

subtracting, $13z=13$

$$\left.\begin{aligned} z&=1,\\ x&=2,\\ y&=3.\end{aligned}\right\}$$

and from (4) $x=2$,

from (1) $y=3$.

Note. After a little practice the student will find that the solution may often be considerably shortened by a suitable combination of the proposed equations. Thus, in the present instance, by adding (1) and (2) and subtracting (3) we obtain $2x-4z=0$, or $x=2z$. Substituting in (1) and (2) we have two easy equations in $y$ and $z$.

*Example* 2. Solve $\frac{x}{2}-1=\frac{y}{6}+1=\frac{z}{7}+2,$

$$\frac{y}{3}+\frac{z}{2}=13.$$

From the equation $\frac{x}{2}-1=\frac{y}{6}+1,$

we have $3x-y=12$ .................................. (1).

Also, from the equation $\frac{x}{2}-1=\frac{z}{7}+2,$

we have $7x-2z=42$ .................................. (2).

And, from the equation $\frac{y}{3}+\frac{z}{2}=13,$

we have $2y+3z=78$ .................................. (3).

Eliminating $z$ from (2) and (3), we have

$$21x+4y=282;$$

and from (1) $12x-4y=48;$

whence $x=10$, $y=18$. Also by substitution in (2) we obtain $z=14$.

*Example* 3. Consider the equations

$5x-3y-z=6$ .......................... (1),

$13x-7y+3z=14$ .......................... (2),

$7x-4y=8$ .......................... (3).

Multiplying (1) by 3 and adding to (2), we have

$$28x-16y=32,$$

or $7x-4y=8.$

Thus the combination of equations (1) and (2) leads us to an equation which is identical with (3), and so to find $x$ and $y$ we have but a single equation $7x-4y=8$, the solution of which is indeterminate. [Art. 100.]

In this and similar cases the anomaly arises from the fact that the equations are not *independent;* in other words, one equation is deducible from the others, and therefore contains no new relation between the unknown quantities which is not already implied in the other equations.

## EXAMPLES XIII. c.

1. $x+2y+2z=11,$
   $2x+y+z=7,$
   $3x+4y+z=14.$

2. $x+3y+4z=14,$
   $x+2y+z=7,$
   $2x+y+2z=2.$

3. $$\begin{aligned} x+4y+3z&=17,\\ 3x+3y+z&=16,\\ 2x+2y+z&=11. \end{aligned}$$

4. $$\begin{aligned} 3x-2y+z&=2,\\ 2x+3y-z&=5,\\ x+y+z&=6. \end{aligned}$$

5. $$\begin{aligned} 2x+y+z&=16,\\ x+2y+z&=9,\\ x+y+2z&=3. \end{aligned}$$

6. $$\begin{aligned} x-2y+3z&=2,\\ 2x-3y+z&=1,\\ 3x-y+2z&=9. \end{aligned}$$

7. $$\begin{aligned} 3x+2y-z&=20,\\ 2x+8y+6z&=70,\\ x-y+6z&=41. \end{aligned}$$

8. $$\begin{aligned} 2x+3y+4z&=20,\\ 3x+4y+5z&=26,\\ 3x+5y+6z&=31. \end{aligned}$$

9. $$\begin{aligned} 3x-4y&=6z-16,\\ 4x-y-z&=5,\\ x&=3y+2(z-1). \end{aligned}$$

10. $$\begin{aligned} 5x+2y&=14,\\ y-6z&=-15,\\ x+2y+z&=0. \end{aligned}$$

11. $$\begin{aligned} x-\frac{y}{5}&=6,\\ y-\frac{z}{7}&=8,\\ z-\frac{x}{2}&=10. \end{aligned}$$

12. $$\frac{y+z}{4}=\frac{z+x}{3}=\frac{x+y}{2},$$
$$x+y+z=27.$$

13. $$\frac{y-z}{3}=\frac{y-x}{2}=5z-4x,$$
$$y+z=2x+1.$$

14. $$\begin{aligned} 2x+3y&=5,\\ 2z-y&=1,\\ 7x-9z&=3. \end{aligned}$$

15. $$\begin{aligned} \frac{1}{2}(x+z-5)=y-z&\\ =2x-11&\\ =9-(x+2z).& \end{aligned}$$

16. $$\begin{aligned} x+20=\frac{3y}{2}+10&\\ =2z+5&\\ =110-(y+z).& \end{aligned}$$

*107. Definition. If the product of two quantities be equal to unity, each is said to be the **reciprocal** of the other. Thus if $ab=1$, $a$ and $b$ are **reciprocals.** They are so called because $a=\frac{1}{b}$, and $b=\frac{1}{a}$; and consequently $a$ is related to $b$ exactly as $b$ is related to $a$.

The reciprocals of $x$ and $y$ are $\frac{1}{x}$ and $\frac{1}{y}$ respectively, and in solving the following equations we consider $\frac{1}{x}$ and $\frac{1}{y}$ as the unknown quantities.

*Example* 1. Solve
$$\frac{8}{x}-\frac{9}{y}=1 \quad\text{.......................}\quad (1),$$
$$\frac{10}{x}+\frac{6}{y}=7 \quad\text{.......................}\quad (2).$$

Multiply (1) by 2 and (2) by 3; thus
$$\frac{16}{x}-\frac{18}{y}=2,$$
$$\frac{30}{x}+\frac{18}{y}=21;$$

adding,
$$\frac{46}{x}=23;$$

multiplying up,
$$46=23x,$$
$$x=2;$$

and by substituting in (1),
$$y=3.$$

*Example* 2. Solve
$$\frac{1}{2x}+\frac{1}{4y}-\frac{1}{3z}=\frac{1}{4} \quad\text{.......................}\quad (1),$$
$$\frac{1}{x}=\frac{1}{3y} \quad\text{.......................}\quad (2),$$
$$\frac{1}{x}-\frac{1}{5y}+\frac{4}{z}=2\frac{2}{15} \quad\text{.......................}\quad (3),$$

clearing of fractional coefficients, we obtain

from (1)
$$\frac{6}{x}+\frac{3}{y}-\frac{4}{z}=3 \quad\text{.......................}\quad (4),$$

from (2)
$$\frac{3}{x}-\frac{1}{y}=0 \quad\text{.......................}\quad (5),$$

from (3)
$$\frac{15}{x}-\frac{3}{y}+\frac{60}{z}=32 \quad\text{.......................}\quad (6).$$

Multiply (4) by 15 and add the result to (6); we have
$$\frac{105}{x}+\frac{42}{y}=77;$$

dividing by 7
$$\frac{15}{x}+\frac{6}{y}=11 \quad\text{.......................}\quad (7);$$

from (5)
$$\frac{18}{x}-\frac{6}{y}=0;$$
$$\therefore \frac{33}{x}=11;$$

$$\therefore \left.\begin{aligned} x&=3,\\ y&=1,\\ z&=2.\end{aligned}\right\}$$

from (5) ($y=1$), from (4) ($z=2$).

## * EXAMPLES XIII. d.

1. $\frac{5}{x}+\frac{6}{y}=3,$ $\frac{15}{x}+\frac{3}{y}=4.$

2. $\frac{6}{x}-\frac{7}{y}=2,$ $\frac{2}{x}+\frac{14}{y}=3.$

3. $\frac{12}{x}-\frac{4}{y}=2,$ $\frac{3}{x}-\frac{2}{y}=0.$

4. $\frac{5}{x}+\frac{16}{y}=79,$ $\frac{16}{x}-\frac{1}{y}=44.$

5. $\frac{21}{x}+\frac{12}{y}=5,$ $\frac{1}{y}-\frac{1}{x}=\frac{1}{42}.$

6. $\frac{5}{x}+\frac{3}{y}=30,$ $\frac{9}{x}-\frac{5}{y}=2.$

7. $\frac{8}{x}-\frac{9}{y}=7,$ $6\left(\frac{1}{x}+\frac{1}{y}\right)=1.$

8. $\frac{25}{x}+\frac{24}{y}=1,$ $20\left(\frac{2}{x}+\frac{3}{y}\right)=7.$

9. $\frac{4}{x}+\frac{27}{y}=42,$ $\frac{14}{x}-\frac{15}{y}=1.$

10. $\frac{3}{x}+\frac{5}{y}=\frac{8}{15},$ $9y-22x=\frac{3xy}{25}.$

11. $\frac{1}{4x}+\frac{1}{3y}=2,$ $\frac{1}{y}-\frac{1}{2x}=1.$

12. $2y-x=4xy,$ $\frac{4}{y}-\frac{3}{x}=9.$

13. $\frac{1}{x}-\frac{2}{y}+4=0,$ $\frac{1}{y}-\frac{1}{z}+1=0,$ $\frac{2}{z}+\frac{3}{x}=14.$

14. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=36,$ $\frac{1}{x}+\frac{3}{y}-\frac{1}{z}=28,$ $\frac{1}{x}+\frac{1}{3y}+\frac{1}{2z}=20.$

15. $\frac{9}{x}-\frac{2}{y}=\frac{5}{z}-\frac{3}{x}=\frac{7}{y}+\frac{15}{2z}=4.$

# CHAPTER XIV.

## PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS.

108. IN the Examples discussed in the last chapter we have seen that it is essential to have as many equations as there are unknown quantities to determine. Consequently in the solution of problems which give rise to simultaneous equations, it will always be necessary that the statement of the question should contain as many independent conditions as there are quantities to be determined.

*Example* 1. Find two numbers whose difference is 11, and one-fifth of whose sum is 9.

Let $x$ be the greater number, $y$ the less;

Then $$x-y=11 \quad\ldots\ldots\ldots\ldots (1),$$

Also $$\frac{x+y}{5}=9,$$

or $$x+y=45 \quad\ldots\ldots\ldots\ldots (2).$$

By addition $2x=56$; and by subtraction $2y=34$.

The numbers are therefore 28 and 17.

*Example* 2. If 15 lbs. of tea and 17 lbs. of coffee together cost £3. 5*s*. 6*d*., and 25 lbs. of tea and 13 lbs. of coffee together cost £4. 6*s*. 2*d*.; find the price of each per pound.

Suppose a pound of tea to cost $x$ shillings,

and . . . . . . . . coffee . . . $y$ . . . . . .

Then from the question, we have

$$15x+17y=65\tfrac{1}{2} \quad\ldots\ldots\ldots\ldots (1),$$

$$25x+13y=86\tfrac{1}{6} \quad\ldots\ldots\ldots\ldots (2).$$

Multiplying (1) by 5 and (2) by 3, we have

$$75x+85y=327\tfrac{1}{2},$$

$$75x+39y=258\tfrac{1}{2}.$$

Subtracting, $$46y=69,$$

$$y=1\tfrac{1}{2}.$$

And from (1) $$15x+25\tfrac{1}{2}=65\tfrac{1}{2},$$

whence $$15x=40;$$

$$\therefore\ x=2\tfrac{2}{3}.$$

$\therefore$ the cost of a pound of tea is $2\frac{2}{3}$ shillings, or 2*s*. 8*d*.,

and the cost of a pound of coffee is $1\frac{1}{2}$ shillings, or 1*s*. 6*d*.

*Example* 3. A person spent 15*s*. 2*d*. in buying oranges at the rate of 3 for twopence and apples at fivepence a dozen; if he had bought five times as many oranges and a quarter of the number of apples he would have spent £2. 4*s*. 2*d*. How many of each did he buy?

Let $x$ be the number of oranges, and $y$ the number of apples.

$$x \text{ oranges cost } \frac{2x}{3} \text{ pence,}$$

$$y \text{ apples cost } \frac{5y}{12} \text{ pence,}$$

$$\therefore \frac{2x}{3} + \frac{5y}{12} = 182 \quad\text{.............................(1).}$$

Again, $5x$ oranges cost $5x \times \frac{2}{3}$, or $\frac{10x}{3}$ pence,

and $\frac{y}{4}$ apples cost $\frac{y}{4} \times \frac{5}{12}$, or $\frac{5y}{48}$ pence,

$$\therefore \frac{10x}{3} + \frac{5y}{48} = 530 \quad\text{.............................(2).}$$

Multiply (1) by 5 and subtract (2) from the result;

$$\therefore \left(\frac{25}{12} - \frac{5}{48}\right) y = 380;$$

$$\therefore \frac{95y}{48} = 380;$$

$$\therefore y = 192,$$

and from (1) $x = 153.$

Thus there were 153 oranges and 192 apples.

*Example* 4. If the numerator of a fraction is increased by 2 and the denominator by 1, it becomes equal to $\frac{5}{8}$; and, if the numerator and denominator are each diminished by 1, it becomes equal to $\frac{1}{2}$: find the fraction.

Let $x$ be the numerator of the fraction, $y$ the denominator; then the fraction is $\frac{x}{y}$.

From the first supposition,

$$\frac{x+2}{y+1} = \frac{5}{8} \quad\text{.............................(1),}$$

from the second,

$$\frac{x-1}{y-1} = \frac{1}{2} \quad\text{.............................(2).}$$

These equations give $x = 8$, $y = 15$.

Thus the fraction is $\frac{8}{15}$.

*Example* 5. The middle digit of a number between 100 and 1000 is zero, and the sum of the other digits is 11. If the digits be reversed, the number so formed exceeds the original number by 495: find it.

Let $x$ be the digit in the units' place,

$y$ . . . . . . . . . . . . . hundreds' place;

then, since the digit in the tens' place is 0, the number will be represented by $100y+x$. [Art. 81, Ex. 4.]

And if the digits are reversed the number so formed will be represented by $100x+y$.

$$\therefore\ 100x+y-(100y+x)=495,$$

or

$$100x+y-100y-x=495;$$

$$\therefore\ 99x-99y=495,$$

that is,

$$x-y=5 \text{ .......................... } (1).$$

Again, since the sum of the digits is 11, and the middle one is 0, we have

$$x+y=11 \text{ .................................. } (2).$$

From (1) and (2) we find $x=8$, $y=3$.

Hence the number is 308.

## EXAMPLES XIV.

1. Find two numbers whose sum is 34, and whose difference is 10.

2. The sum of two numbers is 73, and their difference is 37; find the numbers.

3. One third of the sum of two numbers is 14, and one half of their difference is 4; find the numbers.

4. One nineteenth of the sum of two numbers is 4, and their difference is 30; find the numbers.

5. Half the sum of two numbers is 20, and three times their difference is 18; find the numbers.

6. Six pounds of tea and eleven pounds of sugar cost £1. 3*s*. 8*d*., and eleven pounds of tea and six pounds of sugar cost £1. 18*s*. 8*d*. Find the cost of tea and sugar per pound.

7. Six horses and seven cows can be bought for £250, and thirteen cows and eleven horses can be bought for £461. What is the value of each animal?

8. $A$, $B$, $C$, $D$ have £290 between them; $A$ has twice as much as $C$, and $B$ has three times as much as $D$; also $C$ and $D$ together have £50 less than $A$. Find how much each has.

9. $A$, $B$, $C$, $D$ have £270 between them; $A$ has three times as much as $C$, and $B$ five times as much as $D$; also $A$ and $B$ together have £50 less than eight times what $C$ has. Find how much each has.

**10.** Four times $B$'s age exceeds $A$'s age by twenty years, and one third of $A$'s age is less than $B$'s age by two years: find their ages.

**11.** One eleventh of $A$'s age is greater by two years than one seventh of $B$'s, and twice $B$'s age is equal to what $A$'s age was thirteen years ago: find their ages.

**12.** In eight hours $A$ walks twelve miles more than $B$ does in seven hours; and in thirteen hours $B$ walks seven miles more than $A$ does in nine hours. How many miles does each walk per hour?

**13.** In eleven hours $C$ walks $12\frac{1}{2}$ miles less than $D$ does in twelve hours; and in five hours $D$ walks $3\frac{1}{4}$ miles less than $C$ does in seven hours. How many miles does each walk per hour?

**14.** Find a fraction such that if 1 be added to its denominator it reduces to $\frac{1}{2}$, and reduces to $\frac{3}{5}$ on adding 2 to its numerator.

**15.** Find a fraction which becomes $\frac{1}{2}$ on subtracting 1 from the numerator and adding 2 to the denominator, and reduces to $\frac{1}{3}$ on subtracting 7 from the numerator and 2 from the denominator.

**16.** If 1 be added to the numerator of a fraction it reduces to $\frac{1}{5}$; if 1 be taken from the denominator it reduces to $\frac{1}{7}$: required the fraction.

**17.** If $\frac{2}{3}$ be added to the numerator of a certain fraction the fraction will be increased by $\frac{1}{21}$, and if $\frac{1}{2}$ be taken from its denominator the fraction becomes $\frac{2}{5}$: find it.

**18.** The sum and the difference of a number of two digits and of the number formed by reversing the digits are 110 and 54 respectively: find the numbers.

**19.** The sum and the difference of a number of two digits and of the number formed by reversing the digits are 143 and 27 respectively: find the numbers.

**20.** A certain number of two digits is three times the sum of its digits, and if 45 be added to it the digits will be reversed: find the number.

**21.** A certain number between 10 and 100 is eight times the sum of its digits, and if 45 be subtracted from it the digits will be reversed: find the number.

**22.** A man has a number of pounds and shillings, and he observes that if the pounds were turned into shillings and the shillings into pounds he would gain £5. 14*s*.; but if the pounds were turned into half sovereigns and the shillings into half-crowns he would lose £1. 13*s*. 6*d*. What sum had he?

**23.** In a bag containing black and white balls, half the number of white is equal to a third of the number of black; and twice the whole number of balls exceeds three times the number of black balls by four. How many balls did the bag contain?

**24.** A number consists of three digits, the right hand one being zero. If the left hand and middle digits be interchanged the number is diminished by 180; if the left hand digit be halved and the middle and right hand digits be interchanged the number is diminished by 454. Find the number.

**25.** The wages of 10 men and 8 boys amount to £1. 17*s.*; if 4 men together receive 1*s.* more than 6 boys, what are the wages of each man and boy?

**26.** A grocer wishes to mix spice at 8*s.* a pound with another sort at 5*s.* a pound to make 60 pounds to be sold at 6*s.* a pound: what quantity of each must he take?

**27.** A traveller walks a certain distance; had he gone half a mile an hour faster, he would have walked it in four-fifths of the time: had he gone half a mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road. Find the distance.

**28.** A man walks 35 miles partly at the rate of 4 miles an hour, and partly at 5; if he had walked at 5 miles an hour when he walked at 4, and vice versâ, he would have covered two miles more in the same time. Find the time he was walking.

**29.** Two persons, 27 miles apart, setting out at the same time are together in 9 hours if they walk in the same direction, but in 3 hours if they walk in opposite directions: find their rates of walking.

**30.** A family, consisting of three adults and five children, spends in food £1. 17*s.* 6*d.* a week. Distress, however, comes when they can afford only £1 per week, and the food of each adult is diminished by one-half, and of each child by one-third. Find the cost per week of an adult and of a child.

**31.** If I lend a sum of money at 6 per cent., the interest for a certain time exceeds the loan by £100; but if I lend it at 3 per cent., for a fourth of the time, the loan exceeds its interest by £425. How much do I lend?

**32.** $A$ takes 3 hours longer than $B$ to walk 30 miles; but if he doubles his pace he takes 2 hours less time than $B$: find their rates of walking.

# CHAPTER XV.

## INVOLUTION.

**109.** DEFINITION. **Involution** is the general name for multiplying an expression by itself so as to find its second, third, fourth, or any other power.

Involution may always be effected by actual multiplication. Here, however, we shall give some rules for writing down at once

(1) any power of a simple expression;
(2) the square and cube of any binomial;
(3) the square of any multinomial.

110. It is evident from the Rule of Signs that

(1) no *even* power of *any* quantity can be *negative;*

(2) any *odd* power of a quantity will have *the same sign* as the quantity itself.

NOTE. It is especially worthy of notice that the *square* of every expression, whether positive or negative, is *positive*.

111. From definition we have, by the rules of multiplication,

$$(a^2)^3 = a^2 \,.\, a^2 \,.\, a^2 = a^{2+2+2} = a^6.$$
$$(-x^3)^2 = (-x^3)(-x^3) = x^{3+3} = x^6.$$
$$(-a^5)^3 = (-a^5)(-a^5)(-a^5) = -a^{5+5+5} = -a^{15}.$$
$$(-3a^3)^4 = (-3)^4(a^3)^4 = 81a^{12}.$$

Hence we obtain a rule for raising a simple expression to any proposed power.

RULE. (1) *Raise the coefficient to the required power by Arithmetic, and prefix the proper sign found by Art.* 35.

(2) *Multiply the index of every factor of the expression by the exponent of the power required.*

*Examples.* (1) $(-2x^2)^5 = -32x^{10}$.

(2) $(-3ab^3)^6 = 729a^6b^{18}$.

(3) $\left(\dfrac{2ab^3}{3x^2y}\right)^4 = \dfrac{16a^4b^{12}}{81x^8y^4}$.

It will be seen that in the last case the numerator and the denominator are operated upon separately.

## EXAMPLES XV. a.

Write down the square of each of the following expressions:

1. $3ab^3$. 2. $a^3c$. 3. $7ab^2$. 4. $11b^2c^3$.
5. $4a^4b^5x^2$. 6. $5x^2y^5$. 7. $-2abc^2$. 8. $-3cx^3$.
9. $4xyz^3$. 10. $-\dfrac{2}{3}a^2b^3$. 11. $\dfrac{2x^2}{3y^3}$. 12. $-\dfrac{4}{3x^2y}$.
13. $-\dfrac{7ab}{3}$. 14. $\dfrac{3a^2b^3}{4c^5x^4}$. 15. $-\dfrac{1}{2xy}$. 16. $-2xy^2$.
17. $\dfrac{5ab^3}{2xy}$. 18. $13c^5x^3$. 19. $-\dfrac{1}{4a^4}$. 20. $-\dfrac{3a^5}{5x^3}$.

Write down the cube of each of the following expressions:

21. $2ab^2$. 22. $3x^3$. 23. $4x^4$. 24. $-3a^3b$.
25. $-5ab^2$. 26. $-b^3c^2x$. 27. $-6a^6$. 28. $-2a^7c^2$.
29. $\dfrac{1}{3y^2}$. 30. $-\dfrac{3x^5}{5a^3}$. 31. $7x^3y^4$. 32. $-\dfrac{2}{3}a^5$.

Write down the value of each of the following expressions:

33. $(3a^2b^3)^4$. 34. $(-a^2x)^6$. 35. $(-2x^3y)^5$. 36. $\left(\dfrac{1}{2a^2}\right)^7$.
37. $\left(\dfrac{3x^4}{2y^3}\right)^5$. 38. $\left(\dfrac{2x^3}{3y}\right)^8$. 39. $\left(-\dfrac{x^3}{3}\right)^7$. 40. $\left(-\dfrac{2x^5}{3a^4}\right)^6$.

112. By multiplication we have

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2+2ab+b^2 \text{..........................(1)},\\ (a-b)^2 &= (a-b)(a-b)\\ &= a^2-2ab+b^2 \text{..........................(2)}.\end{aligned}$$

These results are embodied in the following rules:

RULE 1. *The square of the sum of two quantities is equal to the sum of their squares increased by twice their product.*

RULE 2. *The square of the difference of two quantities is equal to the sum of their squares diminished by twice their product.*

*Example* 1. $(x+2y)^2 = x^2 + 2 . x . 2y + (2y)^2$
$= x^2 + 4xy + 4y^2.$

*Example* 2. $(2a^3 - 3b^2)^2 = (2a^3)^2 - 2 . 2a^3 . 3b^2 + (3b^2)^2$
$= 4a^6 - 12a^3b^2 + 9b^4.$

113. These rules may sometimes be conveniently applied to find the squares of numerical quantities.

*Example* 1. The square of $1012 = (1000+12)^2$
$= (1000)^2 + 2 . 1000 . 12 + (12)^2$
$= 1000000 + 24000 + 144$
$= 1024144.$

*Example* 2. The square of $98 = (100-2)^2$
$= (100)^2 - 2 . 100 . 2 + (2)^2$
$= 10000 - 400 + 4$
$= 9604.$

The work is considerably shortened by the omission of the first two steps.

114. We may now extend the rules of Art. 112 thus:

$$\begin{aligned}(a+b+c)^2 &= \{(a+b)+c\}^2\\ &= (a+b)^2 + 2(a+b)c + c^2 \qquad \text{[Art. 112. Rule 1.]}\\ &= a^2+b^2+c^2+2ab+2ac+2bc.\end{aligned}$$

In the same way we may prove

$$(a-b+c)^2 = a^2+b^2+c^2-2ab+2ac-2bc$$
$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

In each of these instances we observe that the square consists of

(1) the sum of the squares of the several terms of the given expression;

(2) twice the sum of the products two and two of the several terms, taken with their proper signs; that is, in each product the sign is + or − according as the quantities composing it have like or unlike signs.

NOTE. The *square terms* are always positive.

The same laws hold whatever be the number of terms in the expression to be squared.

RULE. *To find the square of any multinomial: to the sum of the squares of the several terms add twice the product (with the proper sign) of each term into each of the terms that follow it.*

*Ex.* 1. $(x-2y-3z)^2=x^2+4y^2+9z^2-2.x.2y-2.x.3z+2.2y.3z$
$=x^2+4y^2+9z^2-4xy-6xz+12yz.$

*Ex.* 2. $(1+2x-3x^2)^2=1+4x^2+9x^4+2.1.2x-2.1.3x^2-2.2x.3x^2$
$=1+4x^2+9x^4+4x-6x^2-12x^3$
$=1+4x-2x^2-12x^3+9x^4,$

by collecting like terms and rearranging.

## EXAMPLES XV. b.

Write down the square of each of the following expressions:

**1.** $a+3b$. **2.** $a-3b$. **3.** $x-5y$. **4.** $2x+3y$.

**5.** $3x-y$. **6.** $3x+5y$. **7.** $9x-2y$. **8.** $5ab-c$.

**9.** $pq-r$. **10.** $x-abc$. **11.** $ax+2by$. **12.** $x^2-1$.

**13.** $a-b-c$. **14.** $a+b-c$. **15.** $a+2b+c$.

**16.** $2a-3b+4c$. **17.** $x^2-y^2-z^2$. **18.** $xy+yz+zx$.

**19.** $3p-2q+4r$. **20.** $x^2-x+1$. **21.** $2x^2+3x-1$.

**22.** $x-y+a-b$. **23.** $2x+3y+a-2b$. **24.** $m-n-p-q$.

**25.** $\frac{1}{2}a-2b+\frac{c}{4}$. **26.** $\frac{a}{3}-3b-\frac{3}{2}$. **27.** $\frac{2}{3}x^2-x+\frac{3}{2}$.

115. By actual multiplication, we have

$$(a+b)^3=(a+b)(a+b)(a+b)$$
$$=a^3+3a^2b+3ab^2+b^3.$$

Also $$(a-b)^3=a^3-3a^2b+3ab^2-b^3.$$

By observing the law of formation of the terms in these results we can write down the cube of any binomial.

*Example* 1. $(2x+y)^3=(2x)^3+3(2x)^2y+3(2x)y^2+y^3$
$=8x^3+12x^2y+6xy^2+y^3.$

*Example* 2. $(3x-2a^2)^3=(3x)^3-3(3x)^2(2a^2)+3(3x)(2a^2)^2-(2a^2)^3$
$=27x^3-54x^2a^2+36xa^4-8a^6.$

## EXAMPLES XV. c.

Write down the cube of each of the following expressions:

**1.** $x+a$. **2.** $x-a$. **3.** $x-2y$. **4.** $2x+y$.

**5.** $3x-5y$. **6.** $ab+c$. **7.** $2ab-3c$. **8.** $5a-bc$.

**9.** $x^2+4y^2$. **10.** $4x^2-5y^2$. **11.** $2a^3-3b^2$. **12.** $5x^5-4y^4$.

**13.** $a-\frac{2b}{3}$. **14.** $\frac{a}{3}+2$. **15.** $\frac{x^2}{3}-3x$. **16.** $\frac{a}{6}+2x$.

# CHAPTER XVI.

## EVOLUTION.

116. THE root of any proposed expression is that quantity which being multiplied by itself the requisite number of times produces the given expression. [Art. 15.]

The operation of finding the root is called **Evolution**: it is the reverse of Involution.

117. By the Rule of Signs we see that

(1) any *even* root of a *positive* quantity may be either *positive* or *negative;*

(2) *no negative* quantity can have an *even* root;

(3) every *odd* root of a quantity has the same sign as the quantity itself.

NOTE. It is especially worthy of notice that every positive quantity has two square roots equal in magnitude, but opposite in sign.

*Example.* $\sqrt{(9a^2x^6)} = \pm 3ax^3$.

In the present Chapter, however, we shall confine our attention to the positive root.

*Examples.*

(1) $\sqrt{(a^6b^4)} = a^3b^2$.

(2) $\sqrt[3]{(-x^9)} = -x^3$.

(3) $\sqrt[5]{(c^{20})} = c^4$.

(4) $\sqrt[4]{(81x^{12})} = 3x^3$.

118. Hence we obtain a general rule for extracting any proposed root of a simple expression:

RULE. (1) *Find the root of the coefficient by Arithmetic, and prefix the proper sign found by* Art. 35.

(2) *Divide the exponent of every factor of the expression by the index of the proposed root.*

*Examples.*

(1) $\sqrt[3]{(-64x^6)} = -4x^2$.

(2) $\sqrt[4]{(16a^8)} = 2a^2$.

(3) $\sqrt{\left(\frac{81x^{10}}{25c^4}\right)} = \frac{9x^5}{5c^2}$.

It will be seen that in the last case we operate separately upon the numerator and the denominator.

## EXAMPLES XVI. a.

Write down the square root of each of the following expressions:

**1.** $4a^2b^4$. **2.** $9x^6y^2$. **3.** $25x^4y^6$. **4.** $16a^4b^2c^6$.

**5.** $81a^6b^8$. **6.** $100x^8$. **7.** $a^{20}b^{16}c^4$. **8.** $a^8b^2c^{12}$.

**9.** $64x^6y^{18}$. **10.** $\dfrac{36}{a^{36}}$. **11.** $\dfrac{a^{16}b^8}{16}$. **12.** $\dfrac{289y^4}{25}$.

**13.** $\dfrac{324x^{12}}{169y^6}$. **14.** $\dfrac{81a^{18}}{36b^{12}}$. **15.** $\dfrac{256x^2y^4}{289p^{14}}$. **16.** $\dfrac{400a^{40}b^{20}}{81x^{10}y^{18}}$.

Write down the cube root of each of the following expressions:

**17.** $27a^6b^3c^3$. **18.** $-8a^{12}b^9$. **19.** $64x^6y^3z^{12}$. **20.** $-343a^{12}b^{18}$.

**21.** $-\dfrac{x^{12}y^9}{125}$. **22.** $\dfrac{8x^9}{729y^{15}}$. **23.** $\dfrac{125a^3b^6}{216x^6y^9}$. **24.** $-\dfrac{27x^{27}}{64y^{63}}$.

Write down the value of each of the following expressions:

**25.** $\sqrt[4]{(a^8x^{12})}$. **26.** $\sqrt[7]{(x^{14}y^{21})}$. **27.** $\sqrt[5]{(32x^5y^{10})}$.

**28.** $\sqrt[6]{(729a^{18}b^6)}$. **29.** $\sqrt[8]{(256a^8x^{64})}$. **30.** $\sqrt[5]{(-x^{10}y^{15})}$.

**31.** $\sqrt[7]{\dfrac{128}{a^{63}b^{56}}}$. **32.** $\sqrt[10]{\dfrac{a^{30}x^{50}}{b^{100}}}$. **33.** $\sqrt[9]{\dfrac{a^{18}}{b^{27}c^{36}}}$.

119. *To find the square root of a compound expression.*

Since the square of $a+b$ is $a^2+2ab+b^2$, we have to discover a process by which $a$ and $b$, the terms of the root, can be found when $a^2+2ab+b^2$ is given.

The first term, $a$, is the square root of $a^2$.

Arrange the terms according to powers of one letter $a$. The first term is $a^2$, and its square root is $a$. Set this down as the first term of the required root. Subtract $a^2$ from the given expression and the remainder is $2ab+b^2$ or $(2a+b)\times b$.

Thus, $b$, the second term of the root, will be the quotient when the remainder is divided by $2a+b$.

This divisor consists of two terms:

1. The double of $a$, the term of the root already found.
2. $b$, the new term itself.

The work may be arranged as follows:

$$\begin{array}{rl}
 & a^2+2ab+b^2\,(a+b \\
 & a^2 \\
\hline
2a+b & \;\;\;\;\;\;\;\; 2ab+b^2 \\
 & \;\;\;\;\;\;\;\; 2ab+b^2 \\
\hline
\end{array}$$

*Example* 1. Find the square root of $9x^2-42xy+49y^2$.

$$\begin{array}{r|l}
 & 9x^2-42xy+49y^2\,(3x-7y \\
 & 9x^2 \\ \hline
6x-7y & -42xy+49y^2 \\
 & -42xy+49y^2 \\ \hline
\end{array}$$

*Explanation.* The square root of $9x^2$ is $3x$, and this is the first term of the root.

By doubling this we obtain $6x$, which is the first term of the divisor. Divide $-42xy$, the first term of the remainder, by $6x$ and we get $-7y$, the new term in the root, which has to be annexed both to the root and divisor. Next multiply the complete divisor by $-7y$ and subtract the result from the first remainder. There is now no remainder and the root has been found.

The process can be extended so as to find the square root of any multinomial. The first two terms of the root will be obtained as before. When we have brought down *the second remainder*, the first part of the new divisor is obtained by doubling the terms of the root already found. We then divide the first term of the remainder by the first term of the new divisor, and set down the result as the next term in the root and in the divisor. We next multiply the complete divisor by the last term of the root and subtract the product from the last remainder. If there is now no remainder the root has been found; if there is a remainder we continue the process.

*Example* 2. Find the square root of

$$25x^2a^2-12xa^3+16x^4+4a^4-24x^3a.$$

Rearrange in descending powers of $x$.

$$\begin{array}{r|l}
 & 16x^4-24x^3a+25x^2a^2-12xa^3+4a^4\,(4x^2-3xa+2a^2 \\
 & 16x^4 \\ \hline
8x^2-3xa & -24x^3a+25x^2a^2 \\
 & -24x^3a+\ \ 9x^2a^2 \\ \hline
8x^2-6xa+2a^2 & \qquad\qquad 16x^2a^2-12xa^3+4a^4 \\
 & \qquad\qquad 16x^2a^2-12xa^3+4a^4 \\ \hline
\end{array}$$

*Explanation.* When we have obtained two terms in the root, $4x^2-3xa$, we have a remainder

$$16x^2a^2-12xa^3+4a^4.$$

Double the terms of the root already found and place the result, $8x^2-6xa$, as the first part of the divisor. Divide $16x^2a^2$, the first term of the remainder, by $8x^2$, the first term of the divisor; we get $+2a^2$ which we annex both to the root and divisor. Now multiply the complete divisor by $2a^2$ and subtract. There is no remainder and the root is found.

## EXAMPLES XVI. b.

Find the square root of each of the following expressions:

**1.** $x^2+4xy+4y^2$.

**2.** $9a^2+12ab+4b^2$.

**3.** $x^2-10xy+25y^2$.

**4.** $4x^2-12xy+9y^2$.

**5.** $81x^2+18xy+y^2$.

**6.** $25x^2-30xy+9y^2$.

**7.** $x^4-2x^2y^2+y^4$.

**8.** $1-2a^3+a^6$.

**9.** $a^4-2a^3+3a^2-2a+1$.

**10.** $4x^4-12x^3+29x^2-30x+25$.

**11.** $9x^4-12x^3-2x^2+4x+1$.

**12.** $x^4-4x^3+6x^2-4x+1$.

**13.** $4a^4+4a^3-7a^2-4a+4$.

**14.** $1-10x+27x^2-10x^3+x^4$.

**15.** $4x^2+9y^2+25z^2+12xy-30yz-20xz$.

**16.** $16x^6+16x^7-4x^8-4x^9+x^{10}$.

**17.** $x^6-22x^4+34x^3+121x^2-374x+289$.

**18.** $25x^4-30ax^3+49a^2x^2-24a^3x+16a^4$.

**19.** $4x^4+4x^2y^2-12x^2z^2+y^4-6y^2z^2+9z^4$.

**20.** $6ab^2c-4a^2bc+a^2b^2+4a^2c^2+9b^2c^2-12abc^2$.

**21.** $-6b^2c^2+9c^4+b^4-12c^2a^2+4a^4+4a^2b^2$.

**22.** $4x^4+9y^4+13x^2y^2-6xy^3-4x^3y$.

**23.** $67x^2+49+9x^4-70x-30x^3$.

**24.** $1-4x+10x^2-20x^3+25x^4-24x^5+16x^6$.

**25.** $6acx^5+4b^2x^4+a^2x^{10}+9c^2-12bcx^2-4abx^7$.

[Arts. 120, 121, and Examples XVI. c. may be postponed until Chap. XXIV. has been read.]

*120. When the expression whose root is required contains fractional terms, we may proceed as before, the fractional part of the work being performed by the rules explained in Chap. XII.

*121. There is one important point to be observed when an expression contains powers of a certain letter and also powers of its reciprocal. Thus in the expression

$$2x+\frac{1}{x^2}+4+x^3+\frac{5}{x}+7x^2+\frac{8}{x^3},$$

the order of *descending* powers is

$$x^3+7x^2+2x+4+\frac{5}{x}+\frac{1}{x^2}+\frac{8}{x^3};$$

and the numerical quantity 4 stands between $x$ and $\frac{1}{x}$.

The reason for this arrangement will appear in Chap. XXX.

*Example.* Find the square root of $24+\frac{16y^2}{x^2}-\frac{8x}{y}+\frac{x^2}{y^2}-\frac{32y}{x}$.

Arrange the expression in descending powers of $y$.

$$\begin{array}{l|l}
 & \frac{16y^2}{x^2}-\frac{32y}{x}+24-\frac{8x}{y}+\frac{x^2}{y^2}\left(\frac{4y}{x}-4+\frac{x}{y}\right. \\
 & \frac{16y^2}{x^2} \\ \hline
\frac{8y}{x}-4 & -\frac{32y}{x}+24 \\
 & -\frac{32y}{x}+16 \\ \hline
\frac{8y}{x}-8+\frac{x}{y} & 8-\frac{8x}{y}+\frac{x^2}{y^2} \\
 & 8-\frac{8x}{y}+\frac{x^2}{y^2} \\ \hline
\end{array}$$

Here the second term in the root, $-4$, arises from division of $-\frac{32y}{x}$ by $\frac{8y}{x}$, and the third term, $\frac{x}{y}$, arises from division of 8 by $\frac{8y}{x}$; thus $8\div\frac{8y}{x}=8\times\frac{x}{8y}=\frac{x}{y}$.

## *EXAMPLES XVI. c.

Find the square root of each of the following expressions:

1. $\frac{x^2}{4}-3x+9$.

2. $4-\frac{4x}{y}+\frac{x^2}{y^2}$.

3. $\frac{x^2}{25}+\frac{2xy}{5}+y^2$.

4. $\frac{x^2}{y^2}+\frac{10x}{y}+25$.

5. $\frac{x^2}{4y^2}-\frac{2x}{y}+4$.

6. $\frac{x^2}{y^2}-\frac{2ax}{by}+\frac{a^2}{b^2}$.

7. $\frac{64x^2}{9y^2}+\frac{32x}{3y}+4$.

8. $\frac{9x^2}{25}-2+\frac{25}{9x^2}$.

9. $\frac{a^4}{64}+\frac{a^3}{8}-a+1$.

10. $x^4+2x^3-x+\frac{1}{4}$.

11. $-3a^3+\frac{25}{9}+a^4-5a+\frac{67}{12}a^2$.

12. $x^4-2x+\frac{1}{9}+\frac{29}{3}x^2-6x^3$.

13. $\frac{a^4}{4}+\frac{a^3}{x}+\frac{a^2}{x^2}-ax-2+\frac{x^2}{a^2}$.

14. $x^4-2x^3+\frac{3x^2}{2}-\frac{x}{2}+\frac{1}{16}$.

15. $\frac{x^4}{4}+4x^2+\frac{ax^2}{3}+\frac{a^2}{9}-2x^3-\frac{4ax}{3}$.

16. $\frac{9a^2}{x^2}-\frac{6a}{5x}+\frac{101}{25}-\frac{4x}{15a}+\frac{4x^2}{9a^2}$.

17. $16m^4+\frac{16}{3}m^2n+8m^2+\frac{4}{9}n^2+\frac{4}{3}n+1$.

18. $4x^4+32x^2+96+\frac{64}{x^4}+\frac{128}{x^2}$.

[The rest of the Chapter may be postponed until Chap. XXVII. has been read.]

*122. *To find the cube root of a compound expression.*

Since the cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$, we have to discover a process by which $a$ and $b$, the terms of the root, can be found when $a^3+3a^2b+3ab^2+b^3$ is given.

The first term $a$ is the cube root of $a^3$.

Arrange the terms according to powers of one letter $a$; then the first term is $a^3$, and its cube root $a$. Set this down as the first term of the required root. Subtract $a^3$ from the given expression and the remainder is

$$3a^2b+3ab^2+b^3 \text{ or } (3a^2+3ab+b^2)\times b.$$

Thus $b$, the second term of the root, will be the quotient when the remainder is divided by $3a^2+3ab+b^2$.

This divisor consists of three terms:

1. Three times the square of $a$, the term of the root already found.

2. Three times the product of this first term $a$, and the new term $b$.

3. The square of $b$.

The work may be arranged as follows:

$$\begin{array}{rl|l}
 & & a^3+3a^2b+3ab^2+b^3\ (a+b \\
 & & a^3 \\ \hline
3\,(a)^2 = & 3a^2 & 3a^2b+3ab^2+b^3 \\
3\times a\times b = & +3ab & \\
(b)^2 = & +b^2 & \\ \cline{2-2}
 & 3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 \\ \cline{3-3}
\end{array}$$

*Example* 1. Find the cube root of $8x^3-36x^2y+54xy^2-27y^3$.

$$\begin{array}{rl|l}
 & & 8x^3-36x^2y+54xy^2-27y^3\ (2x-3y \\
 & & 8x^3 \\ \hline
3\,(2x)^2 = & 12x^2 & -36x^2y+54xy^2-27y^3 \\
3\times 2x\times(-3y) = & -18xy & \\
(-3y)^2 = & +9y^2 & \\ \cline{2-2}
 & 12x^2-18xy+9y^2 & -36x^2y+54xy^2-27y^3 \\ \cline{3-3}
\end{array}$$

*Example* 2. Find the cube root of $27+108x+90x^2-80x^3-60x^4+48x^5-8x^6$.

$$
\begin{array}{lll}
 & & 27+108x+90x^2-80x^3-60x^4+48x^5-8x^6\,(3+4x-2x^2 \\
 & & 27 \\
3\times(3)^2 = 27 & & \quad 108x+90x^2-80x^3 \\
3\times 3 \times 4x = \quad +36x & & \\
(4x)^2 = \qquad\quad +16x^2 & & \\
\overline{27+36x+16x^2} & & \quad \underline{108x+144x^2+64x^3} \\
3\times(3+4x)^2 = 27+72x+48x^2 & & \qquad -54x^2-144x^3-60x^4+48x^5-8x^6 \\
3\times(3+4x)\times(-2x^2) = \quad -18x^2-24x^3 & & \\
(-2x^2)^2 = \qquad\qquad +4x^4 & & \\
\overline{27+72x+30x^2-24x^3+4x^4} & & \qquad \underline{-54x^2-144x^3-60x^4+48x^5-8x^6}
\end{array}
$$

*Explanation.* When we have obtained two terms in the root, $3+4x$, we have a remainder

$$-54x^2-144x^3-60x^4+48x^5-8x^6.$$

Take 3 times the square of the root already found and place the result, $27+72x+48x^2$, as the first part of the new divisor. Divide $-54x^2$, the first term of the remainder, by 27, the first term of the divisor; this gives a new term of the root $-2x^2$. To complete the divisor we take 3 times the product of $(3+4x)$ and $-2x^2$, and also the square of $-2x^2$. Now multiply the complete divisor by $-2x^2$ and subtract; there is no remainder and the root is found.

## *EXAMPLES XVI. d.

Find the cube root of each of the following expressions:

**1.** $a^3+3a^2+3a+1$. **2.** $x^3+6x^2+12x+8$.

**3.** $a^3x^3-3a^2x^2y^2+3axy^4-y^6$. **4.** $8m^3-12m^2+6m-1$.

**5.** $64a^3-144a^2b+108ab^2-27b^3$.

**6.** $1+3x+6x^2+7x^3+6x^4+3x^5+x^6$.

**7.** $1-6x+21x^2-44x^3+63x^4-54x^5+27x^6$.

**8.** $a^3+6a^2b-3a^2c+12ab^2-12abc+3ac^2+8b^3-12b^2c+6bc^2-c^3$.

**9.** $8a^6-36a^5+66a^4-63a^3+33a^2-9a+1$.

**10.** $y^6-3y^5+6y^4-7y^3+6y^2-3y+1$.

**11.** $8x^6+12x^5-30x^4-35x^3+45x^2+27x-27$.

**12.** $27x^6-54x^5a+117x^4a^2-116x^3a^3+117x^2a^4-54xa^5+27a^6$.

**13.** $27x^6-27x^5-18x^4+17x^3+6x^2-3x-1$.

**14.** $24x^4y^2+96x^2y^4-6x^5y+x^6-96xy^5+64y^6-56x^3y^3$.

**15.** $216+342x^2+171x^4+27x^6-27x^5-109x^3-108x$.

*123. We add some examples of cube root where fractional terms occur in the given expressions.

*Example.* Find the cube root of $54-27x^3+\frac{8}{x^6}-\frac{36}{x^3}$.

Arrange the expression in *ascending* powers of $x$.

$$\begin{array}{llr|l}
 & & & \frac{8}{x^6}-\frac{36}{x^3}+54-27x^3\left(\frac{2}{x^2}-3x\right. \\
 & & & \frac{8}{x^6} \\
\hline
3\times\left(\frac{2}{x^2}\right)^2 & = & \frac{12}{x^4} & -\frac{36}{x^3}+54-27x^3 \\
3\times\frac{2}{x^2}\times(-3x) & = & -\frac{18}{x} & \\
(-3x)^2 & = & +9x^2 & \\
\cline{3-3}
 & & \frac{12}{x^4}-\frac{18}{x}+9x^2 & -\frac{36}{x^3}+54-27x^3 \\
\cline{4-4}
\end{array}$$

## *EXAMPLES XVI. e.

Find the cube root of each of the following expressions:

1. $\frac{x^3}{8}-\frac{3x^2}{4}+\frac{3x}{2}-1.$

2. $\frac{x^3}{27}+\frac{2x^2}{3}+4x+8.$

3. $8x^3-4x^2y^2+\frac{2}{3}xy^4-\frac{y^6}{27}.$

4. $\frac{27x^3}{64y^3}-\frac{27x^2}{8y^2}+9x-8.$

5. $x^3-9x+\frac{27}{x}-\frac{27}{x^3}.$

6. $\frac{x^6}{y^3}-6x^4+12x^2y^3-8y^6.$

7. $\frac{x^3}{y^3}+\frac{6x^2}{y^2}+\frac{9x}{y}-4-\frac{9y}{x}+\frac{6y^2}{x^2}-\frac{y^3}{x^3}.$

8. $\frac{x^3}{27}-\frac{x^2}{3}+2x-7+\frac{18}{x}-\frac{27}{x^2}+\frac{27}{x^3}.$

9. $\frac{x^3}{a^3}-\frac{12x^2}{a^2}+\frac{54x}{a}-112+\frac{108a}{x}-\frac{48a^2}{x^2}+\frac{8a^3}{x^3}.$

10. $\frac{64a^3}{x^3}-\frac{192a^2}{x^2}+\frac{240a}{x}-160+\frac{60x}{a}-\frac{12x^2}{a^2}+\frac{x^3}{a^3}.$

11. $\frac{6b}{a}+\frac{6a}{b}-7+\frac{a^3}{b^3}-\frac{3a^2}{b^2}-\frac{3b^2}{a^2}+\frac{b^3}{a^3}.$

12. $\frac{60x^4}{y^4}-\frac{80x^3}{y^3}-\frac{90x^2}{y^2}+\frac{8x^6}{y^6}+\frac{108x}{y}-27+\frac{48x^5}{y^5}.$

*124. Before leaving the subject of Evolution it may be useful to remark that the ordinary rules for extracting square and cube roots in Arithmetic are based upon the Algebraical methods we have explained in the present Chapter.

*Example* 1. Find the square root of 5329.

Since 5329 lies between 4900 and 6400, that is between $(70)^2$ and $(80)^2$, its square root consists of two figures and lies between 70 and 80. Hence, corresponding to $a$ the first term of the root in the Algebraical process of Art. 119, we here have 70.

The analogy between the Algebraical and Arithmetical methods will be seen by comparing the cases we give below.

$$\begin{array}{r|l} & a^2+2ab+b^2\,(a+b \\ & a^2 \\ \hline 2a+b & 2ab+b^2 \\ & 2ab+b^2 \\ \hline \end{array} \qquad\qquad \begin{array}{r|l} & 5329\,(70+3=73. \\ & 4900 \\ \hline 140+3=143 & 429 \\ & 429 \\ \hline \end{array}$$

*Example* 2. Find the square root of 53824.

Here 53824 lies between 40000 and 90000, that is between $(200)^2$ and $(300)^2$.

$$
\begin{array}{r|l}
\multicolumn{1}{r}{} & \qquad\ \ a \qquad b \quad c \\
\multicolumn{1}{r}{} & 53824\,(200+30+2=232. \\
\multicolumn{1}{r}{} & 40000 \\
\hline
2a+b\ldots\ldots400+30=430 & 13824 \\
 & 12900 \\
\hline
2(a+b)+c\ldots\ldots460+2=462 & 924 \\
 & 924 \\
\hline
\end{array}
$$

*Example* 3. Find the cube root of 614125.

Since 614125 lies between 512000 and 729000, that is between $(80)^3$ and $(90)^3$, therefore its cube root consists of two figures and lies between 80 and 90.

$$
\begin{array}{r|l}
\multicolumn{1}{r}{} & \qquad\ \ a+b \\
\multicolumn{1}{r}{} & 614125\,(80+5=85. \\
\multicolumn{1}{r}{} & 512000 \\
\hline
3a^2=3\times(80)^2\ =19200 & 102125 \\
3\times a\times b=3\times80\times5=\ \ 1200 & \\
b^2=\qquad\ 5\times5=\quad\ 25 & \\
\overline{20425} & 102125 \\
\hline
\end{array}
$$

In Arithmetic the ciphers are usually omitted and there are also several other modifications of the Algebraical rules. It would be out of place to discuss these fully here: some reference will be made to them in Chap. XXIX.

# CHAPTER XVII.

## RESOLUTION INTO FACTORS.

125. DEFINITION. When an algebraical expression is the product of two or more expressions each of these latter quantities is called a **factor** of it, and the determination of these quantities is called the **resolution** of the expression into its factors.

In this chapter we shall explain the principal rules by which the resolution of expressions into their component factors may be effected.

126. When each of the terms which compose an expression is divisible by a common factor, the expression may be simplified by dividing each term separately by this factor, and enclosing the quotient within brackets; the common factor being placed outside as a coefficient.

*Example* 1. The terms of the expression $3a^2-6ab$ have a common factor $3a$;

$$\therefore\ 3a^2-6ab=3a(a-2b).$$

*Example* 2. $5a^2bx^3-15abx^2-20b^3x^2=5bx^2(a^2x-3a-4b^2)$.

## EXAMPLES XVII. a.

Resolve into factors:

**1.** $a^3-ax$. **2.** $x^3-x^2$. **3.** $2a-2a^2$. **4.** $a^2-ab^2$.
**5.** $7p^2+p$. **6.** $8x-2x^2$. **7.** $5ax-5a^3x^2$.
**8.** $3x^2+x^5$. **9.** $x^2+xy$. **10.** $x^3-x^2y$. **11.** $5x-25x^2y$.
**12.** $15+25x^2$. **13.** $16x+64x^2y$. **14.** $15a^2-225a^4$.
**15.** $54-81x$. **16.** $10x^3-25x^4y$. **17.** $3x^3-x^2+x$.
**18.** $6x^3+2x^4+4x^5$. **19.** $x^3-x^2y+xy^2$.
**20.** $3a^4-3a^3b+6a^2b^2$. **21.** $2x^2y^3-6x^3y^2+2xy^3$.
**22.** $6x^3-9x^2y+12xy^2$. **23.** $5x^5-10a^2x^3-15a^3x^3$.
**24.** $7a-7a^3+14a^4$. **25.** $38a^3x^5+57a^4x^2$.

127. Expressions which contain four terms may sometimes be resolved into factors by arranging the terms in suitable pairs.

*Example* 1. Resolve into factors $x^2-ax+bx-ab$.

Noticing that the first two terms contain a factor $x$, and the last two terms a factor $b$, we enclose the first two terms in one bracket, and the last two in another. Thus,

$$\begin{aligned} x^2-ax+bx-ab &= (x^2-ax)+(bx-ab) \\ &= x(x-a)+b(x-a) \qquad\ldots\ldots\ldots\ldots(1) \\ &= (x-a)(x+b), \end{aligned}$$

since each bracket of (1) contains the same factor $x-a$.

*Example* 2. Resolve into factors $6x^2-9ax+4bx-6ab$.

$$\begin{aligned} 6x^2-9ax+4bx-6ab &= (6x^2-9ax)+(4bx-6ab) \\ &= 3x(2x-3a)+2b(2x-3a) \\ &= (2x-3a)(3x+2b). \end{aligned}$$

*Example* 3. Resolve into factors $12a^2-4ab-3ax^2+bx^2$.

$$\begin{aligned} 12a^2-4ab-3ax^2+bx^2 &= (12a^2-4ab)-(3ax^2-bx^2) \\ &= 4a(3a-b)-x^2(3a-b) \\ &= (3a-b)(4a-x^2). \end{aligned}$$

NOTE. In the first line of work it is usually sufficient to see that each pair contains some common factor. Any suitably chosen pairs will bring out the same result. Thus, in the last example, by a different arrangement, we have

$$\begin{aligned} 12a^2-4ab-3ax^2+bx^2 &= (12a^2-3ax^2)-(4ab-bx^2) \\ &= 3a(4a-x^2)-b(4a-x^2) \\ &= (4a-x^2)(3a-b). \end{aligned}$$

The same result as before, for the order in which the factors of a product is written is of course immaterial.

## EXAMPLES XVII. b.

Resolve into factors:

**1.** $a^2+ab+ac+bc$.

**2.** $a^2-ac+ab-bc$.

**3.** $a^2c^2+acd+abc+bd$.

**4.** $a^2+3a+ac+3c$.

**5.** $2x+cx+2c+c^2$.

**6.** $x^2-ax+5x-5a$.

**7.** $5a+ab+5b+b^2$.

**8.** $ab-by-ay+y^2$.

**9.** $ax-bx-az+bz$.

**10.** $pr+qr-ps-qs$.

**11.** $mx-my-nx+ny$.

**12.** $mx-ma+nx-na$.

**13.** $2ax+ay+2bx+by$.

**14.** $3ax-bx-3ay+by$.

**15.** $6x^2+3xy-2ax-ay.$

**16.** $mx-2my-nx+2ny.$

**17.** $ax^2-3bxy-axy+3by^2.$

**18.** $x^2+mxy-4xy-4my^2.$

**19.** $ax^2+bx^2+2a+2b.$

**20.** $x^2-3x-xy+3y.$

**21.** $2x^4-x^3+4x-2.$

**22.** $3x^3+5x^2+3x+5.$

**23.** $x^4+x^3+2x+2.$

**24.** $y^3-y^2+y-1.$

**25.** $axy+bcxy-az-bcz.$

**26.** $f^2x^2+g^2x^2-ag^2-af^2.$

**27.** $2ax^2+3axy-2bxy-3by^2.$

**28.** $amx^2+bmxy-anxy-bny^2.$

**29.** $ax-bx+by+cy-cx-ay.$

**30.** $a^2x+abx+ac+aby+b^2y+bc.$

128. Before proceeding to the next case of resolution into factors the student is advised to refer to Chap. v. Art. 44. Attention has there been drawn to the way in which, in forming the product of two binomials, the coefficients of the different terms combine so as to give a trinomial result. Thus, by Art. 44,

$$(x+5)(x+3)=x^2+8x+15 \dots\dots\dots\dots (1),$$
$$(x-5)(x-3)=x^2-8x+15 \dots\dots\dots\dots (2),$$
$$(x+5)(x-3)=x^2+2x-15 \dots\dots\dots\dots (3),$$
$$(x-5)(x+3)=x^2-2x-15 \dots\dots\dots\dots (4).$$

We now propose to consider the converse problem: namely, the resolution of a trinomial expression, similar to those which occur on the right-hand side of the above identities, into its component binomial factors.

By examining the above results, we notice that:

1. The first term of both the factors is $x$.

2. The product of the second terms of the two factors is equal to the third term of the trinomial; e.g. in (2) above we see that 15 is the product of $-5$ and $-3$; while in (3) $-15$ is the product of $+5$ and $-3$.

3. The algebraic sum of the second terms of the two factors is equal to the coefficient of $x$ in the trinomial; e.g. in (4) the sum of $-5$ and $+3$ gives $-2$, the coefficient of $x$ in the trinomial.

The application of these laws will be easily understood from the following examples.

*Example* 1. Resolve into factors $x^2+11x+24$.

The second terms of the factors must be such that their product is $+24$, and their sum $+11$. It is clear that they must be $+8$ and $+3$.

$$\therefore\ x^2+11x+24=(x+8)(x+3).$$

*Example* 2. Resolve into factors $x^2-10x+24$.

The second terms of the factors must be such that their product is $+24$, and their sum $-10$. Hence they must *both* be *negative*, and it is easy to see that they must be $-6$ and $-4$.

$$\therefore\ x^2-10x+24=(x-6)(x-4).$$

*Example* 3.
$$\begin{aligned} x^2-18x+81&=(x-9)(x-9)\\ &=(x-9)^2. \end{aligned}$$

*Example* 4.
$$\begin{aligned} x^4+10x^2+25&=(x^2+5)(x^2+5)\\ &=(x^2+5)^2. \end{aligned}$$

*Example* 5. Resolve into factors $x^2-11ax+10a^2$.

The second terms of the factors must be such that their product is $+10a^2$, and their sum $-11a$. Hence they must be $-10a$ and $-a$.

$$\therefore\ x^2-11ax+10a^2=(x-10a)(x-a).$$

**Note.** In examples of this kind the student should always verify his results, by forming the product (*mentally*, as explained in Chap. v.) of the factors he has chosen.

## EXAMPLES XVII. c.

Resolve into factors :

**1.** $a^2+3a+2$. **2.** $a^2+2a+1$. **3.** $a^2+7a+12$.
**4.** $a^2-7a+12$. **5.** $x^2-11x+30$. **6.** $x^2-15x+56$.
**7.** $x^2-19x+90$. **8.** $x^2+13x+42$. **9.** $x^2-21x+110$.
**10.** $x^2-21x+108$. **11.** $x^2-21x+80$. **12.** $x^2+21x+90$.
**13.** $x^2-19x+84$. **14.** $x^2-19x+78$. **15.** $x^2-18x+45$.
**16.** $x^2+20x+96$. **17.** $x^2-26x+165$. **18.** $x^2-21x+104$.
**19.** $x^2+23x+102$. **20.** $a^2-24a+95$. **21.** $a^2-32a+256$.
**22.** $a^2+30a+225$. **23.** $a^2+54a+729$. **24.** $a^2-38a+361$.
**25.** $a^2-14ab+49b^2$. **26.** $a^2+5ab+6b^2$.
**27.** $m^2-13mn+40n^2$. **28.** $m^2-22mn+105n^2$.
**29.** $x^2-23xy+132y^2$. **30.** $x^2-26xy+169y^2$.
**31.** $x^4+8x^2+7$. **32.** $x^4+9x^2y^2+14y^4$.
**33.** $x^2y^2-16xy+39$. **34.** $x^2+49xy+600y^2$.
**35.** $x^2y^2+34xy+289$. **36.** $a^4b^4+37a^2b^2+300$.
**37.** $a^2-20abx+75b^2x^2$. **38.** $x^2+43xy+390y^2$.
**39.** $a^2-29ab+54b^2$. **40.** $x^4+162x^2+6561$.
**41.** $12-7x+x^2$. **42.** $20+9x+x^2$.
**43.** $132-23x+x^2$. **44.** $88+19x+x^2$.
**45.** $130+31xy+x^2y^2$. **46.** $143-24xa+x^2a^2$.
**47.** $204-29x^2+x^4$. **48.** $216+35x+x^2$.

129. Next consider a case where the third term of the trinomial is negative.

*Example* 1. Resolve into factors $x^2+2x-35$.

The second terms of the factors must be such that their product is $-35$, and their *algebraical sum* $+2$. Hence they must have *opposite* signs, and the greater of them must be *positive* in order to give its sign to their sum.

The required terms are therefore $+7$ and $-5$.

$$\therefore\ x^2+2x-35=(x+7)(x-5).$$

*Example* 2. Resolve into factors $x^2-3x-54$.

The second terms of the factors must be such that their product is $-54$, and their *algebraical sum* $-3$. Hence they must have *opposite* signs, and the greater of them must be *negative* in order to give its sign to their sum.

The required terms are therefore $-9$ and $+6$.

$$\therefore\ x^2-3x-54=(x-9)(x+6).$$

Remembering that in these cases the numerical quantities *must have opposite signs*, if preferred, the following method may be adopted.

*Example* 3. Resolve into factors $x^2y^2+23xy-420$.

Find two numbers whose product is 420, and whose *difference* is 23. These are 35 and 12; hence inserting the signs so that the positive may predominate, we have

$$x^2y^2+23xy-420=(xy+35)(xy-12).$$

## EXAMPLES XVII. d.

Resolve into factors:

**1.** $x^2-x-2$,
**2.** $x^2+x-2$.
**3.** $x^2-x-6$.
**4.** $x^2+x-6$.
**5.** $x^2-2x-3$.
**6.** $x^2+2x-3$.
**7.** $x^2+x-56$.
**8.** $x^2+3x-40$.
**9.** $x^2-4x-12$.
**10.** $a^2-a-20$.
**11.** $a^2-4a-21$.
**12.** $a^2+a-20$.
**13.** $a^2-4a-117$.
**14.** $x^2+9x-36$.
**15.** $x^2+x-156$.
**16.** $x^2+x-110$.
**17.** $x^2-9x-90$.
**18.** $x^2-x-240$.
**19.** $a^2-12a-85$.
**20.** $a^2-11a-152$.
**21.** $x^2y^2-5xy-24$.
**22.** $x^2+7xy-60y^2$.
**23.** $x^2+ax-42a^2$.
**24.** $x^2-32xy-105y^2$.
**25.** $a^2-ay-210y^2$.
**26.** $x^2+18x-115$.
**27.** $x^2-20xy-96y^2$.

**28.** $x^2+16x-260.$ **29.** $a^2-11a-26.$ **30.** $a^2y^2+14ay-240.$

**31.** $a^4-a^2b^2-56b^4.$ **32.** $x^4-14x^2-51.$ **33.** $y^4+6x^2y^2-27x^4.$

**34.** $a^2b^2-3abc-10c^2.$ **35.** $a^2+12abx-28b^2x^2.$

**36.** $a^2-18axy-243x^2y^2.$ **37.** $x^4+13a^2x^2-300a^4.$

**38.** $x^4-a^2x^2-132a^4.$ **39.** $x^4-a^2x^2-462a^4.$

**40.** $x^6+x^3-870.$ **41.** $2+x-x^2.$ **42.** $6+x-x^2.$

**43.** $110-x-x^2.$ **44.** $380-x-x^2.$ **45.** $120-7ax-a^2x^2.$

**46.** $65+8xy-x^2y^2.$ **47.** $98-7x-x^2.$ **48.** $204-5x-x^2.$

130. We proceed now to the resolution into factors of trinomial expressions when the coefficient of the highest power is not unity.

Again, referring to Chap. v. Art. 44, we may write down the following results:

$$(3x+2)(x+4)=3x^2+14x+8\ldots\ldots\ldots\ldots(1),$$
$$(3x-2)(x-4)=3x^2-14x+8\ldots\ldots\ldots\ldots(2),$$
$$(3x+2)(x-4)=3x^2-10x-8\ldots\ldots\ldots\ldots(3),$$
$$(3x-2)(x+4)=3x^2+10x-8\ldots\ldots\ldots\ldots(4).$$

The converse problem presents more difficulty than the cases we have yet considered.

Before endeavouring to give a general method of procedure it will be worth while to examine in detail two of the identities given above.

Consider the result $3x^2-14x+8=(3x-2)(x-4)$.

The first term $3x^2$ is the product of $3x$ and $x$.

The third term $+8$ .................... $-2$ and $-4$.

The middle term $-14x$ is the result of adding together the two products $3x\times -4$ and $x\times -2$.

Again, consider the result $3x^2-10x-8=(3x+2)(x-4)$.

The first term $3x^2$ is the product of $3x$ and $x$.

The third term $-8$ .................... $+2$ and $-4$.

The middle term $-10x$ is the result of adding together the two products $3x\times -4$ and $x\times 2$; and its sign is negative because the greater of these two products is negative.

131. The beginner will frequently find that it is not easy to select the proper factors at the first trial. Practice alone will enable him to detect at a glance whether any pair he has chosen will combine so as to give the correct coefficients of the expression to be resolved.

*Example.* Resolve into factors $7x^2 - 19x - 6$.

Write down $(7x \quad 3)(x \quad 2)$ for a first trial, noticing that 3 and 2 must have opposite signs. These factors give $7x^2$ and $-6$ for the first and third terms. But since $7 \times 2 - 3 \times 1 = 11$, the combination fails to give the correct coefficient of the middle term.

Next try $(7x \quad 2)(x \quad 3)$.

Since $7 \times 3 - 2 \times 1 = 19$, these factors will be correct if we insert the signs so that the negative shall predominate.

Thus $$7x^2 - 19x - 6 = (7x + 2)(x - 3).$$

[Verify by mental multiplication.]

132. In actual work it will not be necessary to put down all these steps at length. The student will soon find that the different cases may be rapidly reviewed, and the unsuitable combinations rejected at once.

It is especially important to pay attention to the two following hints:

1. If the third term of the binomial is positive, then the second terms of its factors have both the same sign, and this sign is the same as that of the middle term of the trinomial.

2. If the third term of the trinomial is negative, then the second terms of its factors have opposite signs.

*Example* 1. Resolve into factors $14x^2 + 29x - 15$ ..............(1),
$14x^2 - 29x - 15$ ..............(2).

In each case we may write down $(7x \quad 3)(2x \quad 5)$ as a first trial, noticing that 3 and 5 must have opposite signs.

And since $7 \times 5 - 3 \times 2 = 29$, we have only now to insert the proper signs in each factor.

In (1) the positive sign must predominate,
in (2) the negative ................

Therefore $$14x^2 + 29x - 15 = (7x - 3)(2x + 5).$$
$$14x^2 - 29x - 15 = (7x + 3)(2x - 5).$$

*Example* 2. Resolve into factors $5x^2 + 17x + 6$ ................. (1),
$5x^2 - 17x + 6$ ................. (2).

In (1) we notice that the factors which give 6 are both positive.
In (2) ...................................... negative.

And therefore for (1) we may write $(5x + \quad)(x + \quad)$.
(2) ........... $(5x - \quad)(x - \quad)$.

And, since $5 \times 3 + 1 \times 2 = 17$, we see that
$$5x^2 + 17x + 6 = (5x + 2)(x + 3).$$
$$5x^2 - 17x + 6 = (5x - 2)(x - 3).$$

**Note.** In each expression the third term 6 also admits of factors 6 and 1; but this is one of the cases referred to above which the student would reject at once as unsuitable.

*Example* 3. $9x^2-48xy+64y^2=(3x-8y)(3x-8y)$
$=(3x-8y)^2.$

*Example* 4. $6+7x-5x^2=(3+5x)(2-x).$

## EXAMPLES XVII. e.

Resolve into factors:

**1.** $2x^2+3x+1.$ **2.** $3x^2+5x+2.$ **3.** $2x^2+5x+2.$
**4.** $3x^2+10x+3.$ **5.** $2x^2+9x+4.$ **6.** $3x^2+8x+4.$
**7.** $2x^2+7x+6.$ **8.** $2x^2+11x+5.$ **9.** $3x^2+11x+6.$
**10.** $5x^2+11x+2.$ **11.** $2x^2+3x-2.$ **12.** $3x^2+x-2.$
**13.** $4x^2+11x-3.$ **14.** $3x^2+14x-5.$ **15.** $2x^2+15x-8.$
**16.** $2x^2-x-1.$ **17.** $3x^2+7x-6.$ **18.** $2x^2+x-28.$
**19.** $3x^2+13x-30.$ **20.** $6x^2+7x-3.$ **21.** $6x^2-7x-3.$
**22.** $3x^2+7x+4.$ **23.** $3x^2+23x+14.$ **24.** $2x^2-x-15.$
**25.** $3x^2+19x-14.$ **26.** $3x^2-19x-14.$ **27.** $6x^2-31x+35.$
**28.** $4x^2+x-14.$ **29.** $3x^2-13x+14.$ **30.** $3x^2+41x+26.$
**31.** $4x^2+23x+15.$ **32.** $2x^2-5xy-3y^2.$ **33.** $8x^2-38x+35.$
**34.** $12x^2-23xy+10y^2.$ **35.** $15x^2+224x-15.$
**36.** $15x^2-77x+10.$ **37.** $12x^2-31x-15.$
**38.** $24x^2+22x-21.$ **39.** $72x^2-145x+72.$
**40.** $24x^2-29xy-4y^2.$ **41.** $2-3x-2x^2.$
**42.** $3+11x-4x^2.$ **43.** $6+5x-6x^2.$ **44.** $4-5x-6x^2.$
**45.** $5+32x-21x^2.$ **46.** $7+10x+3x^2.$ **47.** $18-33x+5x^2.$
**48.** $8+6x-5x^2.$ **49.** $20-9x-20x^2.$ **50.** $24+37x-72x^2.$

133. By multiplying $a+b$ by $a-b$ we obtain the identity

$$(a+b)(a-b)=a^2-b^2,$$

a result which may be verbally expressed as follows:

*The product of the sum and the difference of any two quantities is equal to the difference of their squares.*

Conversely, *the difference of the squares of any two quantities is equal to the product of the sum and the difference of the two quantities.*

Thus any expression which is the difference of two squares may at once be resolved into factors.

*Example*, Resolve into factors $25x^2-16y^2$.

$$25x^2-16y^2=(5x)^2-(4y)^2.$$

Therefore the first factor is the sum of $5x$ and $4y$,
and the second factor is the difference of $5x$ and $4y$.

$$\therefore\ 25x^2-16y^2=(5x+4y)(5x-4y).$$

The intermediate steps may usually be omitted.

*Example.* $1-49c^6=(1+7c^3)(1-7c^3)$.

The difference of the squares of two numerical quantities is sometimes conveniently found by the aid of the formula

$$a^2-b^2=(a+b)(a-b).$$

*Example.*

$$\begin{aligned}(329)^2-(171)^2&=(329+171)(329-171)\\&=500\times 158\\&=79000.\end{aligned}$$

## EXAMPLES XVII. f.

Resolve into factors:

**1.** $x^2-4$. **2.** $a^2-81$. **3.** $y^2-100$. **4.** $c^2-144$.
**5.** $9-a^2$. **6.** $49-c^2$. **7.** $121-x^2$. **8.** $400-a^2$.
**9.** $x^2-9a^2$. **10.** $y^2-25x^2$. **11.** $36x^2-25b^2$. **12.** $9x^2-1$.
**13.** $36p^2-49q^2$. **14.** $4k^2-1$. **15.** $49-100k^2$.
**16.** $1-25x^2$. **17.** $a^2-4b^2$. **18.** $9x^2-y^2$.
**19.** $p^2q^2-36$. **20.** $a^2b^2-4c^2d^2$. **21.** $x^4-9$.
**22.** $9a^4-121$. **23.** $25x^2-64$. **24.** $81a^4-49x^4$.
**25.** $x^6-25$. **26.** $1-36a^6$. **27.** $9x^4-a^2$.
**28.** $81x^6-25a^2$. **29.** $x^4a^2-49$. **30.** $a^2-64x^6$.
**31.** $a^2b^2-9x^6$. **32.** $x^6y^6-4$. **33.** $1-a^2b^2$.
**34.** $4-x^2$. **35.** $9-4a^2$. **36.** $9a^4-25b^4$.
**37.** $x^4-16b^2$. **38.** $x^2-25y^2$. **39.** $1-100b^2$.
**40.** $25-64x^2$. **41.** $121a^2-81x^2$. **42.** $p^2q^2-64a^4$.
**43.** $64x^2-25z^6$. **44.** $49x^4-16y^4$. **45.** $81p^4z^6-25b^2$.
**46.** $16x^{16}-9y^6$. **47.** $36x^{36}-49a^{14}$. **48.** $1-100a^6b^4c^2$.
**49.** $25x^{10}-16a^8$. **50.** $a^2b^4c^6-x^{10}$.

Find by resolving into factors the value of

**51.** $(575)^2-(425)^2$. **52.** $(121)^2-(120)^2$. **53.** $(750)^2-(250)^2$.
**54.** $(339)^2-(319)^2$. **55.** $(753)^2-(253)^2$. **56.** $(101)^2-(99)^2$.
**57.** $(1723)^2-(277)^2$. **58.** $(1639)^2-(739)^2$.
**59.** $(1811)^2-(689)^2$. **60.** $(2731)^2-(269)^2$.
**61.** $(8133)^2-(8131)^2$. **62.** $(10001)^2-1$.

134. When one or both of the squares is a compound quantity the same method is employed.

*Example* 1. Resolve into factors $(a+2b)^2-16x^2$.

The sum of $a+2b$ and $4x$ is $a+2b+4x$,

and their difference is $a+2b-4x$.

$$\therefore\ (a+2b)^2-16x^2=(a+2b+4x)(a+2b-4x).$$

*Example* 2. Resolve into factors $x^2-(2b-3c)^2$.

The sum of $x$ and $2b-3c$ is $x+2b-3c$,

and their difference is $x-(2b-3c)=x-2b+3c$.

$$\therefore\ x^2-(2b-3c)^2=(x+2b-3c)(x-2b+3c).$$

If the factors contain like terms they should be collected so as to give the result in its simplest form.

*Example* 3. $(3x+7y)^2-(2x-3y)^2$

$$\begin{aligned}&=\{(3x+7y)+(2x-3y)\}\{(3x+7y)-(2x-3y)\}\\&=(3x+7y+2x-3y)(3x+7y-2x+3y)\\&=(5x+4y)(x+10y).\end{aligned}$$

## EXAMPLES XVII. g.

Resolve into factors:

**1.** $(a+b)^2-c^2$.
**2.** $(a-b)^2-c^2$.
**3.** $(x+y)^2-4z^2$.
**4.** $(x+2y)^2-a^2$.
**5.** $(a+3b)^2-16x^2$.
**6.** $(x+5a)^2-9y^2$.
**7.** $(x+5c)^2-1$.
**8.** $(a-2x)^2-b^2$.
**9.** $(2x-3a)^2-9c^2$.
**10.** $a^2-(b-c)^2$.
**11.** $x^2-(y+z)^2$.
**12.** $4a^2-(y-z)^2$.
**13.** $9x^2-(2a-3b)^2$.
**14.** $1-(a-b)^2$.
**15.** $c^2-(5a-3b)^2$.
**16.** $(a+b)^2-(c+d)^2$.
**17.** $(a-b)^2-(x+y)^2$.
**18.** $(7x+y)^2-1$.
**19.** $(a+b)^2-(m-n)^2$.
**20.** $(a-n)^2-(b+m)^2$.
**21.** $(b-c)^2-(a-x)^2$.
**22.** $(4a+x)^2-(b+y)^2$.
**23.** $(a+2b)^2-(3x+4y)^2$.
**24.** $1-(7a-3b)^2$.
**25.** $(a-b)^2-(x-y)^2$.
**26.** $(a-3x)^2-16y^2$.
**27.** $(2a-5x)^2-1$.
**28.** $(a+b-c)^2-(x-y+z)^2$.
**29.** $(3a+2b)^2-(c+x-2y)^2$.

Resolve into factors and simplify:

**30.** $(x+y)^2-x^2$.
**31.** $x^2-(y-x)^2$.
**32.** $(x+3y)^2-4y^2$.
**33.** $(24x+y)^2-(23x-y)^2$.
**34.** $(5x+2y)^2-(3x-y)^2$.
**35.** $9x^2-(3x-5y)^2$.
**36.** $(7x+3)^2-(5x-4)^2$.

**37.** $(3a+1)^2-(2a-1)^2$.

**38.** $16a^2-(3a+1)^2$.

**39.** $(2a+b-c)^2-(a-b+c)^2$.

**40.** $(x-7y+z)^2-(7y-z)^2$.

**41.** $(x+y-8)^2-(x-8)^2$.

**42.** $(2x+a-3)^2-(3-2x)^2$.

135. By suitably grouping together the terms, compound expressions can often be expressed as the difference of two squares, and so be resolved into factors.

*Example* 1. Resolve into factors $a^2-2ax+x^2-4b^2$.

$$\begin{aligned} a^2-2ax+x^2-4b^2 &= (a^2-2ax+x^2)-4b^2 \\ &= (a-x)^2-(2b)^2 \\ &= (a-x+2b)(a-x-2b). \end{aligned}$$

*Example* 2. Resolve into factors $9a^2-c^2+4cx-4x^2$.

$$\begin{aligned} 9a^2-c^2+4cx-4x^2 &= 9a^2-(c^2-4cx+4x^2) \\ &= (3a)^2-(c-2x)^2 \\ &= (3a+c-2x)(3a-c+2x). \end{aligned}$$

*Example* 3. Resolve into factors $12xy+25-4x^2-9y^2$.

$$\begin{aligned} 12xy+25-4x^2-9y^2 &= 25-(4x^2-12xy+9y^2) \\ &= (5)^2-(2x-3y)^2 \\ &= (5+2x-3y)(5-2x+3y). \end{aligned}$$

*Example* 4. Resolve into factors $2bd-a^2-c^2+b^2+d^2+2ac$.

Here the terms $2bd$ and $2ac$ suggest the proper preliminary arrangement of the expression. Thus

$$\begin{aligned} 2bd-a^2-c^2+b^2+d^2+2ac &= b^2+2bd+d^2-a^2+2ac-c^2 \\ &= b^2+2bd+d^2-(a^2-2ac+c^2) \\ &= (b+d)^2-(a-c)^2 \\ &= (b+d+a-c)(b+d-a+c). \end{aligned}$$

## EXAMPLES XVII. h.

Resolve into factors:

**1.** $x^2+2xy+y^2-a^2$.

**2.** $a^2-2ab+b^2-x^2$.

**3.** $x^2-6ax+9a^2-16b^2$.

**4.** $4a^2+4ab+b^2-9c^2$.

**5.** $x^2+a^2+2ax-y^2$.

**6.** $2ay+a^2+y^2-x^2$.

**7.** $x^2-a^2-2ab-b^2$.

**8.** $y^2-c^2+2cx-x^2$.

**9.** $1-x^2-2xy-y^2$.

**10.** $c^2-x^2-y^2+2xy$.

**11.** $x^2+y^2+2xy-4x^2y^2$.

**12.** $a^2-4ab+4b^2-9a^2c^2$.

**13.** $x^2+2xy+y^2-a^2-2ab-b^2$.

**14.** $a^2-2ab+b^2-c^2-2cd-d^2$.

15. $x^2-4ax+4a^2-b^2+2by-y^2$.
16. $y^2+2by+b^2-a^2-6ax-9x^2$.
17. $x^2-2x+1-a^2-4ab-4b^2$.
18. $9a^2-6a+1-x^2-8dx-16d^2$.
19. $x^2-a^2+y^2-b^2-2xy+2ab$.
20. $a^2+b^2-2ab-c^2-d^2-2cd$.
21. $4x^2-12ax-c^2-k^2-2ck+9a^2$.
22. $a^2+6bx-9b^2x^2-10ab-1+25b^2$.
23. $a^4-25x^6+8a^2x^2-9+30x^3+16x^4$.
24. $x^4-x^2-9-2a^2x^2+a^4+6x$.

136. If we divide $a^3+b^3$ by $a+b$ the quotient is $a^2-ab+b^2$; and if we divide $a^3-b^3$ by $a-b$ the quotient is $a^2+ab+b^2$.

We have therefore the following identities:

$$a^3+b^3=(a+b)(a^2-ab+b^2);$$
$$a^3-b^3=(a-b)(a^2+ab+b^2).$$

These results are very important, and enable us to resolve into factors any expression which can be written as the sum or the difference of two cubes.

*Example* 1. $$\begin{aligned}8x^3-27y^3&=(2x)^3-(3y)^3\\&=(2x-3y)(4x^2+6xy+9y^2).\end{aligned}$$

NOTE. The middle term $6xy$ is the *product* of $2x$ and $3y$.

*Example* 2. $$\begin{aligned}64a^3+1&=(4a)^3+(1)^3\\&=(4a+1)(16a^2-4a+1).\end{aligned}$$

We may usually omit the intermediate step and write down the factors at once.

*Examples.*

(1) $343a^6-27x^3=(7a^2-3x)(49a^4+21a^2x+9x^2)$.
(2) $8x^9+729=(2x^3+9)(4x^6-18x^3+81)$.

## EXAMPLES XVII. k.

Resolve into factors:

1. $x^3-y^3$. 2. $x^3+y^3$. 3. $x^3-1$. 4. $1+a^3$.
5. $8x^3-y^3$. 6. $x^3+8y^3$. 7. $27x^3+1$. 8. $1-8y^3$.
9. $a^3b^3-c^3$. 10. $8x^3+27y^3$. 11. $1-343x^3$. 12. $64+y^3$.
13. $125+a^3$. 14. $216-a^3$. 15. $a^3b^3+512$. 16. $1000y^3-1$.

**17.** $x^3+64y^3$.
**18.** $27-1000x^3$.
**19.** $a^3b^3+216c^3$.
**20.** $343-8x^3$.
**21.** $a^3+27b^3$.
**22.** $27x^3-64y^3$.
**23.** $125x^3-1$.
**24.** $216p^3-343$.
**25.** $x^3y^3+z^3$.
**26.** $a^3b^3c^3-1$.
**27.** $343x^3+1000y^3$.
**28.** $729a^3-64b^3$.
**29.** $8a^3b^3+125x^3$.
**30.** $x^3y^3-216z^3$.
**31.** $x^6-27y^3$.
**32.** $64x^6+125y^3$.
**33.** $8x^3-z^6$.
**34.** $216x^6-b^3$.
**35.** $a^3+343b^3$.
**36.** $a^6+729b^3$.
**37.** $8x^3-729y^6$.
**38.** $p^3q^3-27x^3$.
**39.** $z^3-64y^6$.
**40.** $x^3y^3-512$.

137. Before concluding this Chapter we shall draw attention to a few miscellaneous cases of resolution into factors.

*Example* 1. Resolve into factors $16a^4-81b^4$.

$$\begin{aligned}16a^4-81b^4&=(4a^2+9b^2)(4a^2-9b^2)\\&=(4a^2+9b^2)(2a+3b)(2a-3b).\end{aligned}$$

*Example* 2. Resolve into factors $x^6-y^6$.

$$\begin{aligned}x^6-y^6&=(x^3+y^3)(x^3-y^3)\\&=(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2).\end{aligned}$$

NOTE. When an expression can be arranged either as the difference of two squares, or as the difference of two cubes, each of the methods explained in Arts. 133, 136 will be applicable. It will, however, be found simplest to first use the rule for resolving into factors the difference of two squares.

In all cases where an expression to be resolved contains a simple factor common to each of its terms, this should be first taken outside a bracket as explained in Art. 126.

*Example* 3. Resolve into factors $28x^4y+64x^3y-60x^2y$.

$$\begin{aligned}28x^4y+64x^3y-60x^2y&=4x^2y(7x^2+16x-15)\\&=4x^2y(7x-5)(x+3).\end{aligned}$$

*Example* 4. Resolve into factors $x^3p^2-8y^3p^2-4x^3q^2+32y^3q^2$.

$$\begin{aligned}\text{The expression}&=p^2(x^3-8y^3)-4q^2(x^3-8y^3)\\&=(x^3-8y^3)(p^2-4q^2)\\&=(x-2y)(x^2+2xy+4y^2)(p+2q)(p-2q).\end{aligned}$$

*Example* 5. Resolve into factors $4x^2-25y^2+2x+5y$.

$$\begin{aligned}4x^2-25y^2+2x+5y&=(2x+5y)(2x-5y)+2x+5y\\&=(2x+5y)(2x-5y+1).\end{aligned}$$

## EXAMPLES XVII. 1.

Resolve into two or more factors:

**1.** $a^2-y^2-2yz-z^2$. **2.** $x^6-y^6z^6$. **3.** $6x^2-x-77$.

**4.** $729y^6-64x^6$. **5.** $x^6-4096$.

**6.** $2mn+2xy+m^2+n^2-x^2-y^2$. **7.** $33x^2-16x-65$.

**8.** $a^4+b^4-c^4-d^4+2a^2b^2-2c^2d^2$. **9.** $m^3x+m^3y-n^3x-n^3y$.

**10.** $(a+b+c)^2-(a-b-c)^2$. **11.** $4+4x+2ay+x^2-a^2-y^2$.

**12.** $x^2-10x-119$. **13.** $a^2-b^2-c^2+d^2-2(ad-bc)$.

**14.** $x^2-a^2+y^2-2xy$. **15.** $a^2+x^2-(y^2+z^2)-2(yz-ax)$.

**16.** $21x^2+82x-39$. **17.** $1-a^2x^2-b^2y^2+2abxy$.

**18.** $c^5d^3-c^2-a^2c^3d^3+a^2$. **19.** $a^2x^6-a^2y^6-b^2x^6+b^2y^6$.

**20.** $x^2-6x-247$. **21.** $a^3x^2-c^3x^2-a^3y^2+c^3y^2$.

**22.** $acx^2-bcx+adx-bd$. **23.** $a^2x-b^2x+a^2y-b^2y$.

**24.** $x^4+4x^2y^2z^2+4y^4z^4$. **25.** $a^3b^3+512$. **26.** $2x^2+17x+35$.

**27.** $500x^2y-20y^3$. **28.** $a^5-8a^2b^3$. **29.** $a^2x^3-16x^3y^2$.

**30.** $b^2+c^2-a^2-2bc$. **31.** $5x^4-15x^3-90x^2$.

**32.** $14a^2x^3-35a^3x^2+14a^4x$. **33.** $x^8-1$.

**34.** $1-(m^2+n^2)+2mn$. **35.** $75x^4-48a^4$. **36.** $5a^4b^4-5ab$.

**37.** $8x^2y+52xy+60y$. **38.** $3x^2y^2+26axy+35a^2$.

**39.** $729a^7b-ab^7$. **40.** $a^8x^6-64a^2y^6$.

**41.** $a^{12}-b^{12}$. **42.** $24x^2y^2-30xy^3-36y^4$.

**43.** $(a+b)^4-1$. **44.** $a^4-(b+c)^4$.

**45.** $(c+d)^3-1$. **46.** $1-(x-y)^3$.

**47.** $250(a-b)^3+2$. **48.** $(c+d)^3+(c-d)^3$.

**49.** $8(x+y)^3-(2x-y)^3$. **50.** $x^2-4y^2+x-2y$.

**51.** $a^2-b^2+a-b$. **52.** $(a+b)^2+a+b$.

**53.** $a^3+b^3+a+b$. **54.** $a^2-9b^2+a+3b$.

**55.** $4(x-y)^3-(x-y)$. **56.** $x^4y-x^2y^3-x^3y^2+xy^4$.

## CHAPTER XVIII.

### HIGHEST COMMON FACTOR.

138. We have already explained how to write down by inspection the highest common factor of two or more *simple* expressions. [See Arts. 89, 90.] An analogous method will enable us readily to find the highest common factor of *compound* expressions which are given as the product of factors, or which can be easily resolved into factors.

*Example* 1. Find the highest common factor of

$$4cx^3 \text{ and } 2cx^3+4c^2x^2.$$

It will be easy to pick out the common factors if the expressions are arranged as follows:

$$4cx^3=4cx^3,$$

$$2cx^3+4c^2x^2=2cx^2(x+2c);$$

therefore the H. C. F. is $2cx^2$.

*Example* 2. Find the highest common factor of

$$3a^2+9ab,\ a^3-9ab^2,\ a^3+6a^2b+9ab^2.$$

Resolving each expression into its factors, we have

$$3a^2+9ab=3a(a+3b),$$

$$a^3-9ab^2=a(a+3b)(a-3b),$$

$$a^3+6a^2b+9ab^2=a(a+3b)(a+3b);$$

therefore the H. C. F. is $a(a+3b)$.

139. When there are two or more expressions containing different powers of the same *compound* factor, the student should be careful to notice that the highest common factor must contain the highest power of the compound factor which is common to all the given expressions.

*Example* 1. The highest common factor of

$$x(a-x)^2,\ a(a-x)^3, \text{ and } 2ax(a-x)^5 \text{ is } (a-x)^2.$$

*Example* 2. Find the highest common factor of

$$ax^2+2a^2x+a^3,\quad 2ax^2-4a^2x-6a^3,\quad 3(ax+a^2)^2.$$

Resolving the expressions into factors, we have

$$\begin{aligned} ax^2+2a^2x+a^3 &= a(x^2+2ax+a^2) \\ &= a(x+a)^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1), \\ 2ax^2-4a^2x-6a^3 &= 2a(x^2-2ax-3a^2) \\ &= 2a(x+a)(x-3a) \ldots\ldots\ldots\ldots (2), \\ 3(ax+a^2)^2 &= 3a^2(x+a)^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3). \end{aligned}$$

Therefore from (1), (2), (3), by inspection, the highest common factor is $a(x+a)$.

## EXAMPLES XVIII. a.

Find the highest common factor of

1. $a^2+ab,\ a^2-b^2.$
2. $(x+y)^2,\ x^2-y^2.$
3. $2x^2-2xy,\ x^3-x^2y.$
4. $6x^2-9xy,\ 4x^2-9y^2.$
5. $x^3+x^2y,\ x^3+y^3.$
6. $a^3b-ab^3,\ a^5b^2-a^2b^5.$
7. $a^3-a^2x,\ a^3-ax^2,\ a^4-ax^3.$
8. $a^2-4x^2,\ a^2+2ax.$
9. $a^2bx+ab^2x,\ a^2b-b^3.$
10. $2x^2y-6xy^2,\ x^2-9y^2.$
11. $a^2-x^2,\ a^2-ax,\ a^2x-ax^2.$
12. $4x^2+2xy,\ 12x^2y-3y^3.$
13. $20x-4,\ 50x^2-2.$
14. $6bx+4by,\ 9cx+6cy.$
15. $x^2+x,\ (x+1)^2,\ x^3+1.$
16. $xy-y,\ x^4y-xy.$
17. $x^2-2xy+y^2,\ (x-y)^3.$
18. $x^3+a^2x,\ x^4-a^4.$
19. $x^3+8y^3,\ x^2+xy-2y^2.$
20. $x^4-27a^3x,\ (x-3a)^2.$
21. $x^2+3x+2,\ x^2-4.$
22. $x^2-x-20,\ x^2-9x+20.$
23. $x^2-18x+45,\ x^2-9.$
24. $2x^2-7x+3,\ 3x^2-7x-6.$
25. $12x^2+x-1,\ 15x^2+8x+1.$
26. $2x^2-x-1,\ 3x^2-x-2.$
27. $c^2x^2-d^2,\ acx^2-bcx+adx-bd.$
28. $x^5-xy^2,\ x^3+x^2y+xy+y^2.$
29. $a^3x-a^2bx-6ab^2x,\ a^2bx^2-4ab^2x^2+3b^3x^2.$
30. $2x^2+9x+4,\ 2x^2+11x+5,\ 2x^2-3x-2.$
31. $3x^4+8x^3+4x^2,\ 3x^5+11x^4+6x^3,\ 3x^4-16x^3-12x^2.$

140. It will often happen that expressions cannot be readily resolved into factors. To find the highest common factor in such cases, we adopt a method analogous to that used in Arithmetic for finding the greatest common measure of two or more numbers.

For convenience we here repeat the definition of Art. 88.

**Definition.** The highest common factor of two or more algebraical expressions is the expression of highest dimensions which divides each of them without remainder.

**Note.** The term *greatest common measure* is sometimes used instead of *highest common factor;* but, strictly speaking, the term *greatest common measure* ought to be confined to arithmetical quantities; for the highest common factor is not necessarily the greatest common measure in all cases, as will appear later. [Art. 145.]

141. We begin by working out examples illustrative of the algebraical process of finding the highest common factor, postponing for the present the complete proof of the rules we use. But we may conveniently *enunciate* two principles, which the student should bear in mind in reading the examples which follow.

I. *If an expression contain a certain factor, any multiple of the expression is divisible by that factor.*

II. *If two expressions have a common factor, it will divide their sum and their difference; and also the sum and the difference of any multiples of them.*

*Example.* Find the highest common factor of

$$4x^3-3x^2-24x-9 \text{ and } 8x^3-2x^2-53x-39.$$

$$\begin{array}{r|r|r|l}
x & 4x^3-3x^2-24x-9 & 8x^3-2x^2-53x-39 & 2 \\
 & 4x^3-5x^2-21x & 8x^3-6x^2-48x-18 & \\
\cline{2-3}
2x & 2x^2-\phantom{0}3x-9 & 4x^2-\phantom{0}5x-21 & 2 \\
 & 2x^2-\phantom{0}6x\phantom{-9} & 4x^2-\phantom{0}6x-18 & \\
\cline{2-3}
3 & 3x-9 & x-3 & \\
 & 3x-9 & & \\
\cline{2-2}
\end{array}$$

Therefore the H. C. F. is $x-3$.

*Explanation.* First arrange the given expressions according to descending or ascending powers of $x$. The expressions so arranged having their first terms of the same order, we take for divisor that whose highest power has the smaller coefficient. Arrange the work in parallel columns as above. When the first remainder $4x^2-5x-21$ is made the divisor we put the quotient $x$ to the *left* of the dividend. Again, when the second remainder $2x^2-3x-9$ is in turn made the divisor, the quotient 2 is placed to the *right;* and so on. As in Arithmetic, the last divisor $x-3$ is the highest common factor required.

142. This method is only useful to determine the *compound* factor of the highest common factor. *Simple* factors of the given

expressions must be first removed from them, and the highest common factor of these, if any, must be observed and multiplied into the *compound* factor given by the rule.

*Example.* Find the highest common factor of

$$24x^4-2x^3-60x^2-32x \text{ and } 18x^4-6x^3-39x^2-18x.$$

We have $24x^4-2x^3-60x^2-32x=2x(12x^3-x^2-30x-16)$,
and $18x^4-6x^3-39x^2-18x=3x(6x^3-2x^2-13x-6)$.

Also $2x$ and $3x$ have the common factor $x$. Removing the simple factors $2x$ and $3x$, and *reserving* their common factor $x$, we continue as in Art. 141.

$$\begin{array}{r|r|r|l}
2x & 6x^3-2x^2-13x-6 & 12x^3-\phantom{4}x^2-30x-16 & 2 \\
 & 6x^3-8x^2-\phantom{1}8x\phantom{-6} & 12x^3-4x^2-26x-12 & \\
\cline{2-3}
2 & 6x^2-5x-6 & 3x^2-4x-4 & x \\
 & 6x^2-8x-8 & 3x^2+2x\phantom{-4} & \\
\cline{2-3}
 & 3x+2 & -6x-4 & -2 \\
 & & -6x-4 & \\
\cline{3-3}
\end{array}$$

Therefore the H. C. F. is $x(3x+2)$.

143. So far the process of Arithmetic has been found exactly applicable to the algebraical expressions we have considered. But in many cases certain modifications of the arithmetical method will be found necessary. These will be more clearly understood if it is remembered that, at every stage of the work, the remainder must contain as a factor of itself the highest common factor we are seeking. [See Art. 141, I & II.]

*Example* 1. Find the highest common factor of

$$3x^3-13x^2+23x-21 \text{ and } 6x^3+x^2-44x+21.$$

$$\begin{array}{r|r|l}
3x^3-13x^2+23x-21 & 6x^3+\phantom{26}x^2-44x+21 & 2 \\
 & 6x^3-26x^2+46x-42 & \\
\cline{2-2}
 & 27x^2-90x+63 & \\
\end{array}$$

Here on making $27x^2-90x+63$ a divisor, we find that it is not contained in $3x^3-13x^2+23x-21$ with an *integral* quotient. But noticing that $27x^2-90x+63$ may be written in the form $9(3x^2-10x+7)$, and also bearing in mind that every remainder in the course of the work contains the H. C. F., we conclude that the H. C. F. we are seeking is contained in $9(3x^2-10x+7)$. But the two original expressions have no *simple* factors, therefore their H. C. F. can have none. We may therefore *reject* the factor 9 and go on with divisor $3x^2-10x+7$. Resuming the work, we have

$$\begin{array}{r|r|r|l}
x & 3x^3-13x^2+23x-21 & 3x^2-10x+7 & x \\
 & \underline{3x^3-10x^2+\phantom{0}7x\phantom{-21}} & \underline{3x^2-\phantom{0}7x\phantom{+7}} & \\
-1 & -\phantom{0}3x^2+16x-21 & -\phantom{0}3x+7 & -1 \\
 & \underline{-\phantom{0}3x^2+10x-\phantom{0}7} & \underline{-\phantom{0}3x+7} & \\
 & 2\,)\,\underline{6x-14} & & \\
 & 3x-\phantom{0}7 & & 
\end{array}$$

Therefore the highest common factor is $3x-7$.

The factor 2 has been removed on the same grounds as the factor 9 above.

*Example* 2. Find the highest common factor of

$$2x^3+\phantom{2}x^2-x-2 \text{ .......................... } (1),$$

and $$3x^3-2x^2+x-2 \text{ .......................... } (2).$$

As the expressions stand we cannot begin to divide one by the other without using a fractional quotient. The difficulty may be obviated by *introducing* a suitable factor, just as in the last case we found it useful to remove a factor when we could no longer proceed with the division in the ordinary way. The given expressions have no common *simple* factor, hence their H. C. F. cannot be affected if we multiply either of them by any simple factor.

Multiply (2) by 2, and use (1) as a divisor:

$$\begin{array}{r|r|r|l}
 & 2x^3+\phantom{00}x^2-\phantom{00}x-\phantom{0}2 & 6x^3-\phantom{00}4x^2+\phantom{0}2x-\phantom{0}4 & 3 \\
 & \underline{7\phantom{x^3+\phantom{00}x^2-\phantom{00}x-\phantom{0}2}} & \underline{6x^3+\phantom{00}3x^2-\phantom{0}3x-\phantom{0}6} & \\
-2x & 14x^3+\phantom{0}7x^2-\phantom{0}7x-14 & -\phantom{00}7x^2+\phantom{0}5x+\phantom{0}2 & \\
 & \underline{14x^3-10x^2-\phantom{0}4x\phantom{-14}} & \underline{17\phantom{x^2+\phantom{0}5x+\phantom{0}2}} & \\
17x & 17x^2-\phantom{0}3x-14 & -119x^2+85x+34 & -7 \\
 & \underline{17x^2-17x\phantom{-14}} & \underline{-119x^2+21x+98} & \\
14 & 14x-14 & 64\,)\,\underline{64x-64} & \\
 & \underline{14x-14} & x-\phantom{0}1 & 
\end{array}$$

Therefore the H. C. F. is $x-1$.

After the first division the factor 7 is introduced because the first remainder $-7x^2+5x+2$ will not divide $2x^3+x^2-x-2$.

At the next stage the factor 17 is introduced for a similar reason, and finally the factor 64 is removed as explained in Example 1.

144. From the last two examples it appears that we may multiply or divide either of the given expressions, or any of the remainders which occur in the course of the work, by any factor which does not divide both of the given expressions.

145. Let the two expressions in Example 2, Art. 143, be written in the form

$$2x^3+x^2-x-2=(x-1)(2x^2+3x+2),$$
$$3x^3-2x^2+x-2=(x-1)(3x^2+x+2).$$

Then their highest common factor is $x-1$, and therefore $2x^2+3x+2$ and $3x^2+x+2$ *have no algebraical common divisor.* If, however, we put $x=6$, then

$$2x^3+x^2-x-2=460,$$

and

$$3x^3-2x^2+x-2=580;$$

and the greatest common measure of 460 and 580 is 20; whereas 5 is the numerical value of $x-1$, the algebraical highest common factor. Thus the numerical values of the algebraical highest common factor and of the arithmetical greatest common measure do not in this case agree.

The reason may be explained as follows: when $x=6$, the expressions $2x^2+3x+2$ and $3x^2+x+2$ become equal to 92 and 116 respectively, and have a common arithmetical factor 4; whereas the expressions have no algebraical common factor.

It will thus often happen that the highest common factor of two expressions, and their numerical greatest common measure, when the letters have particular values, are not the same; for this reason the term greatest common measure is inappropriate when applied to algebraical quantities.

## EXAMPLES XVIII. b.

Find the highest common factor of the following expressions:

1. $x^3+2x^2-13x+10$, $x^3+x^2-10x+8$.
2. $x^3-5x^2-99x+40$, $x^3-6x^2-86x+35$.
3. $x^3+2x^2-8x-16$, $x^3+3x^2-8x-24$.
4. $x^3+4x^2-5x-20$, $x^3+6x^2-5x-30$.
5. $x^3-x^2-5x-3$, $x^3-4x^2-11x-6$.
6. $x^3+3x^2-8x-24$, $x^3+3x^2-3x-9$.
7. $a^3-5a^2x+7ax^2-3x^3$, $a^3-3ax^2+2x^3$.
8. $x^3-x^2+x+3$, $x^4+x^3-3x^2-x+2$.
9. $2x^3-5x^2+11x+7$, $4x^3-11x^2+25x+7$.
10. $2x^3+4x^2-7x-14$, $6x^3-10x^2-21x+35$.
11. $3x^4-3x^3-2x^2-x-1$, $6x^4-3x^3-x^2-x-1$.
12. $2x^4-2x^3+x^2+3x-6$, $4x^4-2x^3+3x-9$.
13. $3x^3-3ax^2+2a^2x-2a^3$, $3x^3+12ax^2+2a^2x+8a^3$.
14. $2x^3-9ax^2+9a^2x-7a^3$, $4x^3-20ax^2+20a^2x-16a^3$.
15. $2x^3+7ax^2+4a^2x-3a^3$, $4x^3+9ax^2-2a^2x-a^3$.
16. $6a^3+13a^2x-9ax^2-10x^3$, $9a^3+12a^2x-11ax^2-10x^3$.

17. $24x^4y+72x^3y^2-6x^2y^3-90xy^4,\ 6x^4y^2+13x^3y^3-4x^2y^4-15xy^5.$

18. $4x^5a^2+10x^4a^3-60x^3a^4+54x^2a^5,\ 24x^5a^3+30x^3a^5-126x^2a^6.$

19. $4x^5+14x^4+20x^3+70x^2,\ 8x^7+28x^6-8x^5-12x^4+56x^3.$

20. $72x^3-12ax^2+72a^2x-420a^3,\ 18x^3+42ax^2-282a^2x+270a^3.$

*146. The statements of Art. 141 may be proved as follows.

I. If $F$ divides $A$ it will also divide $mA$.

For suppose $A=aF$, then $mA=maF$.
Thus $F$ is a factor of $mA$.

II. If $F$ divides $A$ and $B$, then it will divide $mA \pm nB$.

For suppose $A=aF$, $B=bF$,

then
$$\begin{aligned} mA \pm nB &= maF \pm nbF \\ &= F(ma \pm nb). \end{aligned}$$

Thus $F$ divides $mA \pm nB$.

*147. We may now enunciate and prove the rule for finding the highest common factor of any two compound algebraical expressions.

We suppose that any simple factors are first removed. [See Example, Art. 142.]

Let $A$ and $B$ be the two expressions after the simple factors have been removed. Let them be arranged in descending or ascending powers of some common letter; also let the highest power of that letter in $B$ be not less than the highest power in $A$.

Divide $B$ by $A$; let $p$ be the quotient, and $C$ the remainder. Suppose $C$ to have a *simple* factor $m$. Remove this factor, and so obtain a new divisor $D$. Further, suppose that in order to make $A$ divisible by $D$ it is necessary to multiply $A$ by a *simple* factor $n$. Let $q$ be the next quotient and $E$ the remainder. Finally, divide $D$ by $E$; let $r$ be the quotient, and suppose that there is no remainder. Then $E$ will be the H.C.F. required.

The work will stand thus:

$$
\begin{array}{rllll}
A) & B & (p & & \\
 & \underline{pA} & & & \\
m) & \underline{C} & & & \\
 & D) & nA & (q & \\
 & & \underline{qD} & & \\
 & & E) & D & (r \\
 & & & \underline{rE} & \\
\end{array}
$$

First, to shew that $E$ is *a* common factor of $A$ and $B$.

By examining the steps of the work, it is clear that $E$ divides $D$, therefore also $qD$; therefore $qD+E$, therefore $nA$; therefore $A$, since $n$ is a *simple* factor.

Again, $E$ divides $D$, therefore $mD$, that is, $C$. And since $E$ divides $A$ and $C$, it also divides $pA+C$, that is, $B$. Hence $E$ divides both $A$ and $B$.

Secondly, to shew that $E$ is the *highest* common factor.

If not, let there be a factor $X$ of higher dimensions than $E$.

Then $X$ divides $A$ and $B$, therefore $B-pA$, that is, $C$; therefore $D$ (since $m$ is a *simple* factor); therefore $nA-qD$, that is, $E$.

Thus $X$ divides $E$; which is impossible since, by hypothesis, $X$ is of higher dimensions than $E$.

Therefore $E$ is the highest common factor.

*148. The highest common factor of three expressions $A, B, C$ may be obtained as follows.

First determine $F$ the highest common factor of $A$ and $B$; next find $G$ the highest common factor of $F$ and $C$; then $G$ will be the required highest common factor of $A$, $B$, $C$.

For $F$ contains *every* factor which is common to $A$ and $B$, and $G$ is the highest common factor of $F$ and $C$. Therefore $G$ is the highest common factor of $A$, $B$, $C$.

# CHAPTER XIX.

## FRACTIONS.

[On first reading the subject, the student may omit the general proofs of the rules given in this Chapter.]

149. IN Chapter XII. we discussed the simpler kinds of fractions, using the ordinary Arithmetical rules. We here propose to give proofs of those rules, and shew that they are applicable to Algebraical fractions.

150. DEFINITION. If a quantity $x$ be divided into $b$ equal parts, and $a$ of these parts be taken, the result is called *the fraction* $\frac{\mathrm{a}}{\mathrm{b}}$ *of* x.

If $x$ be the unit, the fraction $\frac{a}{b}$ of $x$ is called simply "the fraction $\frac{a}{b}$"; so that *the fraction* $\frac{\mathrm{a}}{\mathrm{b}}$ *represents* a *equal parts*, b *of which make up the unit.*

151. *To prove that* $\frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{ma}}{\mathrm{mb}}$, *where* a, b, m *are positive integers.*

By $\frac{a}{b}$ we mean $a$ equal parts, $b$ of which make up the unit ... (1);

by $\frac{ma}{mb}$ . . . . . $ma$ . . . . . . . . . . $mb$ . . . . . . . . . . . . . . . . . . . . . .... (2).

But $\qquad b$ parts in $(1) = mb$ parts in (2);

$\therefore$ 1 part . . . . . $= m$ . . . . . . . . . .

$\therefore$ $a$ parts . . . . $= ma$ . . . . . . . . .

that is,
$$\frac{a}{b} = \frac{ma}{mb}.$$

Conversely,
$$\frac{ma}{mb} = \frac{a}{b}.$$

Hence we have the following rule:

Rule I. *The value of a fraction is not altered if we multiply or divide the numerator and denominator by the same quantity.*

An algebraical fraction may therefore be reduced to an equivalent fraction by dividing numerator and denominator by any common factor; if this factor be the highest common factor, the resulting fraction is said to be in its *lowest terms.*

*Example* 1. Reduce to lowest terms $\frac{24a^3c^2x^2}{18a^3x^2-12a^2x^3}$.

$$\frac{24a^3c^2x^2}{18a^3x^2-12a^2x^3}=\frac{24a^3c^2x^2}{6a^2x^2(3a-2x)}$$
$$=\frac{4ac^2}{3a-2x}.$$

*Example* 2. Reduce to lowest terms $\frac{6x^2-8xy}{9xy-12y^2}$.

$$\frac{6x^2-8xy}{9xy-12y^2}=\frac{2x(3x-4y)}{3y(3x-4y)}=\frac{2x}{3y}.$$

Note. The beginner should be careful not to begin cancelling until he has expressed both numerator and denominator in the most convenient form, by resolution into factors where necessary.

## EXAMPLES XIX. a.

Reduce to lowest terms:

1. $\frac{3a^2-6ab}{2a^2b-4ab^2}$.
2. $\frac{abx+bx^2}{acx+cx^2}$.
3. $\frac{ax}{a^2x^2-ax}$.
4. $\frac{15a^2b^2c}{100(a^3-a^2b)}$.
5. $\frac{4x^2-9y^2}{4x^2+6xy}$.
6. $\frac{20(x^3-y^3)}{5x^2+5xy+5y^2}$.
7. $\frac{x(2a^2-3ax)}{a(4a^2x-9x^3)}$.
8. $\frac{x^3-2xy^2}{x^4-4x^2y^2+4y^4}$.
9. $\frac{(xy-3y^2)^2}{x^3y^2-27y^5}$.
10. $\frac{x^2-5x}{x^2-4x-5}$.
11. $\frac{3x^2+6x}{x^2+4x+4}$.
12. $\frac{5a^3b+10a^2b^2}{3a^2b^2+6ab^3}$.
13. $\frac{x^3y+2x^2y+4xy}{x^3-8}$.
14. $\frac{3a^4+9a^3b+6a^2b^2}{a^4+a^3b-2a^2b^2}$.
15. $\frac{x^4-14x^2-51}{x^4-2x^2-15}$.
16. $\frac{x^2+xy-2y^2}{x^3-y^3}$.
17. $\frac{2x^2+17x+21}{3x^2+26x+35}$.
18. $\frac{a^2x^2-16a^2}{ax^2+9ax+20a}$.
19. $\frac{3x^2+23x+14}{3x^2+41x+26}$.
20. $\frac{27a+a^4}{18a-6a^2+2a^3}$.

152. When the factors of the numerator and denominator cannot be determined by inspection, the fraction may be reduced to its lowest terms by dividing both numerator and denominator by the highest common factor, which may be found by the rules given in Chap. XVIII.

*Example.* Reduce to lowest terms $\frac{3x^3-13x^2+23x-21}{15x^3-38x^2-2x+21}$.

*First Method.* The H.C.F. of numerator and denominator is $3x-7$.

Dividing numerator and denominator by $3x-7$, we obtain as respective quotients $x^2-2x+3$ and $5x^2-x-3$.

$$\text{Thus}\quad \frac{3x^3-13x^2+23x-21}{15x^3-38x^2-2x+21}=\frac{(3x-7)(x^2-2x+3)}{(3x-7)(5x^2-x-3)}=\frac{x^2-2x+3}{5x^2-x-3}.$$

This is the simplest solution for the beginner; but in this and similar cases we may often effect the reduction without actually going through the process of finding the highest common factor.

*Second Method.* By Art. 141, the H.C.F. of numerator and denominator must be a factor of their sum $18x^3-51x^2+21x$, that is, of $3x(3x-7)(2x-1)$. If there be a common divisor it must clearly be $3x-7$; hence arranging numerator and denominator so as to shew $3x-7$ as a factor,

$$\begin{aligned}\text{the fraction} &= \frac{x^2(3x-7)-2x(3x-7)+3(3x-7)}{5x^2(3x-7)-x(3x-7)-3(3x-7)}\\ &=\frac{(3x-7)(x^2-2x+3)}{(3x-7)(5x^2-x-3)}\\ &=\frac{x^2-2x+3}{5x^2-x-3}.\end{aligned}$$

153. If either numerator or denominator can readily be resolved into factors we may use the following method.

*Example.* Reduce to lowest terms $\frac{x^3+3x^2-4x}{7x^3-18x^2+6x+5}$.

The numerator $=x(x^2+3x-4)=x(x+4)(x-1)$.

Of these factors the only one which can be a common divisor is $x-1$. Hence, arranging the denominator,

$$\begin{aligned}\text{the fraction} &= \frac{x(x+4)(x-1)}{7x^2(x-1)-11x(x-1)-5(x-1)}\\ &=\frac{x(x+4)(x-1)}{(x-1)(7x^2-11x-5)}=\frac{x(x+4)}{7x^2-11x-5}.\end{aligned}$$

## EXAMPLES XIX. b.

Reduce to lowest terms:

1. $\dfrac{a^3-a^2b-ab^2-2b^3}{a^3+3a^2b+3ab^2+2b^3}$. 2. $\dfrac{x^3-5x^2+7x-3}{x^3-3x+2}$.

3. $\dfrac{a^3+2a^2-13a+10}{a^3+a^2-10a+8}$. 4. $\dfrac{2x^3+5x^2y-30xy^2+27y^3}{4x^3+5xy^2-21y^3}$.

5. $\dfrac{4a^3+12a^2b-ab^2-15b^3}{6a^3+13a^2b-4ab^2-15b^3}$. 6. $\dfrac{1+2x^2+x^3+2x^4}{1+3x^2+2x^3+3x^4}$.

7. $\dfrac{x^2-2x+1}{3x^3+7x-10}$. 8. $\dfrac{3a^3-3a^2b+ab^2-b^3}{4a^2-5ab+b^2}$.

9. $\dfrac{4x^3+3ax^2+a^3}{x^4+ax^3+a^3x+a^4}$. 10. $\dfrac{4x^3-10x^2+4x+2}{3x^4-2x^3-3x+2}$.

11. $\dfrac{16x^4-72x^2a^2+81a^4}{4x^2+12ax+9a^2}$. 12. $\dfrac{6x^3+x^2-5x-2}{6x^3+5x^2-3x-2}$.

13. $\dfrac{5x^3+2x^2-15x-6}{7x^3-4x^2-21x+12}$. 14. $\dfrac{4x^4+11x^2+25}{4x^4-9x^2+30x-25}$.

15. $\dfrac{3x^3-27ax^2+78a^2x-72a^3}{2x^3+10ax^2-4a^2x-48a^3}$. 16. $\dfrac{ax^3-5a^2x^2-99a^3x+40a^4}{x^4-6ax^3-86a^2x^2+35a^3x}$.

### MULTIPLICATION AND DIVISION OF FRACTIONS.

154. RULE II. *To multiply a fraction by an integer: multiply the numerator by that integer; or, if the denominator be divisible by the integer, divide the denominator by it.*

The rule may be proved as follows:

(1) $\dfrac{a}{b}$ represents $a$ equal parts, $b$ of which make up the unit;

$\dfrac{ac}{b}$ represents $ac$ equal parts, $b$ of which make up the unit;

and the number of parts taken in the second fraction is $c$ times the number taken in the first;

that is, $$\frac{a}{b}\times c=\frac{ac}{b}.$$

(2) $$\frac{a}{bd}\times d=\frac{ad}{bd}, \text{ by the preceding case,}$$

$$=\frac{a}{b}. \qquad \text{[Rule I, Art. 151.]}$$

155. Rule III. *To divide a fraction by an integer: divide the numerator, if it be divisible, by the integer; or, if the numerator be not divisible, multiply the denominator by that integer.*

The rule may be proved as follows:

(1) $\frac{ac}{b}$ represents $ac$ equal parts, $b$ of which make up the unit;

$\frac{a}{b}$ represents $a$ equal parts, $b$ of which make up the unit.

The number of parts taken in the first fraction is $c$ times the number taken in the second. Therefore the second fraction is the quotient of the first fraction divided by $c$;

that is, $$\frac{ac}{b} \div c = \frac{a}{b}.$$

(2) But if the numerator be not divisible by $c$, we have

$$\frac{a}{b} = \frac{ac}{bc};$$

$$\therefore\ \frac{a}{b} \div c = \frac{ac}{bc} \div c$$

$$= \frac{a}{bc}, \text{ by the preceding case.}$$

156. *To find the value of* $\frac{\mathrm{a}}{\mathrm{b}}$ *of* $\frac{\mathrm{c}}{\mathrm{d}}$.

By definition, [Art. 150.]

$\frac{a}{b}$ of $\frac{c}{d}$ represents $a$ equal parts, $b$ of which make up $\frac{c}{d}$.

Since $$b \text{ parts} = \frac{c}{d},$$

$$\text{one part} = \frac{c}{d} \div b = \frac{c}{bd}. \text{ [Rule III, Art. 155.]}$$

$$\therefore\ a \text{ parts} = \frac{c}{bd} \times a = \frac{ac}{bd}; \text{ [Rule II, Art. 154.]}$$

$$\therefore\ \frac{a}{b} \text{ of } \frac{c}{d} = \frac{ac}{bd}.$$

Similarly, $$\frac{c}{d} \text{ of } \frac{a}{b} = \frac{ca}{db}.$$

Therefore $$\frac{a}{b} \text{ of } \frac{c}{d} = \frac{c}{d} \text{ of } \frac{a}{b}.$$

157. Rule IV. *To multiply together two or more fractions: multiply the numerators for a new numerator, and the denominator for a new denominator.*

The definition of multiplication, in the strict sense of the word, supposes that a quantity is to be added to itself a certain number of times. But when the multiplier is a fraction this definition ceases to be intelligible; the operation can therefore be only understood in some extended sense. To find a meaning for $\frac{a}{b} \times \frac{c}{d}$, we may consider that to obtain $\frac{c}{d}$ times $\frac{a}{b}$ we must perform on $\frac{a}{b}$ an operation similar to that we perform on the unit to obtain $\frac{c}{d}$. In this case we take $\frac{c}{d}$ of 1; hence in the former we take $\frac{c}{d}$ of $\frac{a}{b}$.

Thus
$$\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \text{ of } \frac{a}{b}$$
$$= \frac{ac}{bd}. \qquad \text{[Art. 156.]}$$

Similarly, $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ace}{bdf}$; and so for any number of fractions.

158. Rule V. *To divide one fraction by another: invert the divisor, and proceed as in multiplication.*

Since division is the inverse of multiplication, we may define the quotient $x$, when $\frac{a}{b}$ is divided by $\frac{c}{d}$, to be such that
$$x \times \frac{c}{d} = \frac{a}{b}.$$

Multiplying by $\frac{d}{c}$, we have $x \times \frac{c}{d} \times \frac{d}{c} = \frac{a}{b} \times \frac{d}{c}$;
$$\therefore x = \frac{ad}{bc}.$$

Hence
$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}, \qquad \text{[Art. 157.]}$$

which proves the rule.

*Example* 1. Simplify $\dfrac{2a^2+3a}{4a^3}\times\dfrac{4a^2-6a}{12a+18}$.

$$\frac{2a^2+3a}{4a^3}\times\frac{4a^2-6a}{12a+18}=\frac{a(2a+3)}{4a^3}\times\frac{2a(2a-3)}{6(2a+3)}$$
$$=\frac{2a-3}{12a},$$

by cancelling those factors which are common to both numerator and denominator.

*Example* 2. Simplify $\dfrac{6x^2-ax-2a^2}{ax-a^2}\times\dfrac{x-a}{9x^2-4a^2}\div\dfrac{2x+a}{3ax+2a^2}$.

$$\text{The expression}=\frac{6x^2-ax-2a^2}{ax-a^2}\times\frac{x-a}{9x^2-4a^2}\times\frac{3ax+2a^2}{2x+a}$$
$$=\frac{(3x-2a)(2x+a)}{a(x-a)}\times\frac{x-a}{(3x+2a)(3x-2a)}\times\frac{a(3x+2a)}{2x+a}$$
$$=1,$$

since all the factors cancel each other.

## EXAMPLES XIX. c.

Simplify

1. $\dfrac{14x^2-7x}{12x^3+24x^2}\div\dfrac{2x-1}{x^2+2x}$.

2. $\dfrac{a^2b^2+3ab}{4a^2-1}\div\dfrac{ab+3}{2a+1}$.

3. $\dfrac{x^2-4a^2}{ax+2a^2}\times\dfrac{2a}{x-2a}$.

4. $\dfrac{a^2-121}{a^2-4}\div\dfrac{a+11}{a+2}$.

5. $\dfrac{16x^2-9a^2}{x^2-4}\times\dfrac{x-2}{4x-3a}$.

6. $\dfrac{25a^2-b^2}{9a^2x^2-4x^2}\times\dfrac{x(3a+2)}{5a+b}$.

7. $\dfrac{x^2+5x+6}{x^2-1}\times\dfrac{x^2-2x-3}{x^2-9}$.

8. $\dfrac{x^2+3x+2}{x^2+9x+20}\times\dfrac{x^2+7x+12}{x^2+5x+6}$.

9. $\dfrac{2x^2+5x+2}{x^2-4}\times\dfrac{x^2+4x}{2x^2+9x+4}$.

10. $\dfrac{2x^2+13x+15}{4x^2-9}\div\dfrac{2x^2+11x+5}{4x^2-1}$.

11. $\dfrac{x^2-14x-15}{x^2-4x-45}\div\dfrac{x^2-12x-45}{x^2-6x-27}$.

12. $\frac{2x^2-x-1}{2x^2+5x+2}\times\frac{4x^2+x-14}{16x^2-49}$.  13. $\frac{b^4-27b}{2b^2+5b}\times\frac{4b^2-25}{2b^2-11b+15}$.

14. $\frac{x^3-6x^2+36x}{x^2-49}\div\frac{x^4+216x}{x^2-x-42}$.

15. $\frac{64p^2q^2-z^4}{x^2-4}\times\frac{(x-2)^2}{8pq+z^2}\div\frac{x^2-4}{(x+2)^2}$.

16. $\frac{x^2-x-20}{x^2-25}\times\frac{x^2-x-2}{x^2+2x-8}\div\frac{x+1}{x^2+5x}$.

17. $\frac{x^2-18x+80}{x^2-5x-50}\times\frac{x^2-6x-7}{x^2-15x+56}\times\frac{x+5}{x-1}$.

18. $\frac{x^2-8x-9}{x^2-17x+72}\times\frac{x^2-25}{x^2-1}\div\frac{x^2+4x-5}{x^2-9x+8}$.

19. $\frac{4x^2+x-14}{6xy-14y}\times\frac{4x^2}{x^2-4}\times\frac{x-2}{4x-7}\div\frac{2x^2+4x}{3x^2-x-14}$.

20. $\frac{x^2+x-2}{x^2-x-20}\times\frac{x^2+5x+4}{x^2-x}\div\left(\frac{x^2+3x+2}{x^2-2x-15}\times\frac{x+3}{x^2}\right)$.

21. $\frac{4x^2-16x+15}{2x^2+3x+1}\times\frac{x^2-6x-7}{2x^2-17x+21}\times\frac{4x^2-1}{4x^2-20x+25}$.

22. $\frac{x^4-8x}{x^2-4x-5}\times\frac{x^2+2x+1}{x^3-x^2-2x}\div\frac{x^2+2x+4}{x-5}$.

23. $\frac{(a+b)^2-c^2}{a^2+ab-ac}\times\frac{a}{(a+c)^2-b^2}\times\frac{(a-b)^2-c^2}{ab-b^2-bc}$.

24. $\frac{a^2+2ab+b^2-c^2}{a^2-b^2-c^2-2bc}\times\frac{a^2-2ac+c^2-b^2}{b^2-2bc+c^2-a^2}$.

25. $\frac{x^2-64}{x^2+24x+128}\times\frac{x^2+12x-64}{x^3-64}\div\frac{x^2-16x+64}{x^2+4x+16}$.

# CHAPTER XX.

## LOWEST COMMON MULTIPLE.

159. IN Chap. XI. we defined the lowest common multiple of two or more algebraical expressions to be the expression of lowest dimensions which is divisible by each of them without remainder. We there shewed how in the case of *simple* expressions the lowest common multiple could be written down by inspection.

The lowest common multiple of compound expressions which are given as the product of factors, or which can be easily resolved into factors, can be readily found by a similar method.

*Example* 1. The lowest common multiple of $6x^2(a-x)^2$, $8a^3(a-x)^3$ and $12ax(a-x)^5$ is $24a^3x^2(a-x)^5$.

For it consists of the product of

(1) the numerical L. C. M. of the coefficients;

(2) the lowest power of each factor which is divisible by every power of that factor occurring in the given expressions.

*Example* 2. Find the lowest common multiple of

$$3a^2+9ab,\ 2a^3-18ab^2,\ a^3+6a^2b+9ab^2.$$

$$\begin{aligned} 3a^2+9ab &= 3a(a+3b), \\ 2a^3-18ab^2 &= 2a(a+3b)(a-3b), \\ a^3+6a^2b+9ab^2 &= a(a+3b)(a+3b) \\ &= a(a+3b)^2. \end{aligned}$$

Therefore the L. C. M. is $6a(a+3b)^2(a-3b)$.

## EXAMPLES XX. a.

Find the lowest common multiple of

**1.** $x,\ x^2+x$.  **2.** $x^2,\ x^2-3x$.  **3.** $3x^2,\ 4x^2+8x$.

**4.** $21x^3,\ 7x^2(x+1)$.  **5.** $x^2-1,\ x^2+x$.  **6.** $a^2+ab,\ ab+b^2$.

**7.** $4x^2y-y,\ 2x^2+x$.  **8.** $6x^2-2x,\ 9x^2-3x$.

**9.** $x^2+2x,\ x^2+3x+2.$ **10.** $x^2-3x+2,\ x^2-1.$
**11.** $x^2+4x+4,\ x^2+5x+6.$ **12.** $x^2-5x+4,\ x^2-6x+8.$
**13.** $x^2-x-6,\ x^2+x-2,\ x^2-4x+3.$
**14.** $x^2+x-20,\ x^2-10x+24,\ x^2-x-30.$
**15.** $x^2+x-42,\ x^2-11x+30,\ x^2+2x-35.$
**16.** $2x^2+3x+1,\ 2x^2+5x+2,\ x^2+3x+2.$
**17.** $3x^2+11x+6,\ 3x^2+8x+4,\ x^2+5x+6.$
**18.** $5x^2+11x+2,\ 5x^2+16x+3,\ x^2+5x+6.$
**19.** $2x^2+3x-2,\ 2x^2+15x-8,\ x^2+10x+16.$
**20.** $3x^2-x-14,\ 3x^2-13x+14,\ x^2-4.$
**21.** $12x^2+3x-42,\ 12x^3+30x^2+12x,\ 32x^2-40x-28.$
**22.** $3x^4+26x^3+35x^2,\ 6x^2+38x-28,\ 27x^3+27x^2-30x.$
**23.** $60x^4+5x^3-5x^2,\ 60x^2y+32xy+4y,\ 40x^3y-2x^2y-2xy.$
**24.** $8x^2-38xy+35y^2,\ 4x^2-xy-5y^2,\ 2x^2-5xy-7y^2.$
**25.** $12x^2-23xy+10y^2,\ 4x^2-9xy+5y^2,\ 3x^2-5xy+2y^2.$
**26.** $6ax^3+7a^2x^2-3a^3x,\ 3a^2x^2+14a^3x-5a^4,\ 6x^2+39ax+15a^2.$
**27.** $4ax^2y^2+11axy^2-3ay^2, 3x^3y^3-7x^2y^3-6xy^3, 24ax^2-22ax+4a.$

160. When the given expressions are such that their factors cannot be determined by inspection, they must be resolved by finding the highest common factor.

*Example.* Find the lowest common multiple of

$$2x^4+x^3-20x^2-7x+24 \text{ and } 2x^4+3x^3-13x^2-7x+15.$$

The highest common factor is $x^2+2x-3$.

By division, we obtain

$$2x^4+x^3-20x^2-7x+24=(x^2+2x-3)\,(2x^2-3x-8).$$
$$2x^4+3x^3-13x^2-7x+15=(x^2+2x-3)\,(2x^2-x-5).$$

Therefore the L. C. M. is $(x^2+2x-3)\,(2x^2-3x-8)\,(2x^2-x-5)$.

*161. We may now give the proof of the rule for finding the lowest common multiple of two compound algebraical expressions.

Let $A$ and $B$ be the two expressions, and $F$ their highest common factor. Also suppose that $a$ and $b$ are the respective quotients when $A$ and $B$ are divided by $F$; then $A=aF$, $B=bF$. Therefore, since $a$ and $b$ have no common factor, the lowest common multiple of $A$ and $B$ is $abF$, by inspection.

*162. There is an important relation between the highest common factor and the lowest common multiple of two expressions which it is desirable to notice.

Let $F$ be the highest common factor, and $X$ the lowest common multiple of $A$ and $B$. Then, as in the preceding Article,

$$A=aF, \quad B=bF,$$

and $$X=abF.$$

Therefore the product $$\begin{aligned} AB&=aF \,.\, bF \\ &=F \,.\, abF \\ &=FX \quad \ldots\ldots\ldots\ldots\ldots\ldots(1). \end{aligned}$$

Hence *the product of two expressions is equal to the product of their highest common factor and lowest common multiple.*

Again, from (1) $X=\dfrac{AB}{F}=\dfrac{A}{F}\times B=\dfrac{B}{F}\times A$;

hence *the lowest common multiple of two expressions may be found by dividing their product by their highest common factor; or by dividing either of them by their highest common factor, and multiplying the quotient by the other.*

*163. The lowest common multiple of three expressions $A, B, C$ may be obtained as follows.

First, find $X$ the L.C.M. of $A$ and $B$. Next find $Y$ the L.C.M. of $X$ and $C$; then $Y$ will be the required L.C.M. of $A, B, C$.

For $Y$ is the expression of lowest dimensions which is divisible by $X$ and $C$, and $X$ is the expression of lowest dimensions divisible by $A$ and $B$. Therefore $Y$ is the expression of lowest dimensions divisible by all three.

## EXAMPLES XX. b.

1. Find the highest common factor and the lowest common multiple of $x^2-5x+6$, $x^2-4$, $x^3-3x-2$.

2. Find the lowest common multiple of
$ab(x^2+1)+x(a^2+b^2)$ and $ab(x^2-1)+x(a^2-b^2)$.

3. Find the lowest common multiple of $xy-bx$, $xy-ay$,
$y^2-3by+2b^2$, $xy-2bx-ay+2ab$, $xy-bx-ay+ab$.

**4.** Find the highest common factor and the lowest common multiple of $x^3+2x^2-3x$, $2x^3+5x^2-3x$.

**5.** Find the lowest common multiple of

$$1-x,\ (1-x^2)^2,\ (1+x)^3.$$

**6.** Find the lowest common multiple of

$$x^2-10x+24,\ x^2-8x+12,\ x^2-6x+8.$$

**7.** Find the highest common factor and the lowest common multiple of $6x^3+x^2-5x-2$, $6x^3+5x^2-3x-2$.

**8.** Find the lowest common multiple of

$$(bc^2-abc)^2,\ b^2(ac^2-a^3),\ a^2c^2+2ac^3+c^4.$$

**9.** Find the lowest common multiple of

$$x^3-y^3,\ x^3y-y^4,\ y^2(x-y)^2,\ x^2+xy+y^2.$$

Also find the highest common factor of the first three expressions.

**10.** Find the highest common factor of

$$6x^2-13x+6,\ 2x^2+5x-12,\ 6x^2-x-12.$$

Also shew that the lowest common multiple is the product of the three quantities divided by the square of the highest common factor.

**11.** Find the lowest common multiple of

$$x^4+ax^3+a^3x+a^4,\ x^4+a^2x^2+a^4.$$

**12.** Find the highest common factor and the lowest common multiple of $3x^3-7x^2y+5xy^2-y^3$, $x^2y+3xy^2-3x^3-y^3$,

$$3x^3+5x^2y+xy^2-y^3.$$

**13.** Find the highest common factor of

$$4x^3-10x^2+4x+2,\ 3x^4-2x^3-3x+2.$$

**14.** Find the lowest common multiple of

$$a^2-b^2,\ a^3-b^3,\ a^3-a^2b-ab^2-2b^3.$$

**15.** Find the highest common factor and the lowest common multiple of $(2x^2-3a^2)\,y+(2a^2-3y^2)\,x$, $(2a^2+3y^2)\,x+(2x^2+3a^2)\,y$.

**16.** Find the highest common factor and the lowest common multiple of $x^3-9x^2+26x-24$, $x^3-12x^2+47x-60$.

**17.** Find the highest common factor of

$$x^3-15ax^2+48a^2x+64a^3,\ x^2-10ax+16a^2.$$

**18.** Find the lowest common multiple of

$$21x\,(xy-y^2)^2,\ 35\,(x^4y^2-x^2y^4),\ 15y\,(x^2+xy)^2.$$

## CHAPTER XXI.

### ADDITION AND SUBTRACTION OF FRACTIONS.

164. HAVING explained the rules for finding the lowest common multiple of any given expressions, we now proceed to shew how the addition and subtraction of fractions may be effected.

165. *To prove* $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$.

We have $\frac{a}{b}=\frac{ad}{bd}$, and $\frac{c}{d}=\frac{bc}{bd}$. [Rule I, Art. 151.]

Thus in each case we divide the unit into $bd$ equal parts, and we take first $ad$ of these parts, and then $bc$ of them; that is, we take $ad+bc$ of the $bd$ parts of the unit; and this is expressed by the fraction $\frac{ad+bc}{bd}$.

$$\therefore \quad \frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}.$$

Similarly,
$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}.$$

166. Here the fractions have been both expressed with a common denominator $bd$. But if $b$ and $d$ have a common factor, the product $bd$ is not the lowest common denominator, and the fraction $\frac{ad+bc}{bd}$ will not be in its lowest terms. To avoid working with fractions which are not in their lowest terms, some modification of the above will be necessary. In practice it will be found advisable to take the *lowest* common denominator, which is the lowest common multiple of the denominators of the given fractions.

**Rule I.** *To reduce fractions to their lowest common denominator: find the L.C.M. of the given denominators, and take it for the common denominator; divide it by the denominator of the first fraction, and multiply the numerator of this fraction by the quotient so obtained; and do the same with all the other given fractions.*

*Example.* Express with lowest common denominator

$$\frac{5x}{2a(x-a)} \text{ and } \frac{4a}{3x(x^2-a^2)}.$$

The lowest common denominator is $6ax(x-a)(x+a)$.

We must therefore multiply the numerators by $3x(x+a)$ and 2 respectively.

Hence the equivalent fractions are

$$\frac{15x^2(x+a)}{6ax(x-a)(x+a)} \text{ and } \frac{8a}{6ax(x-a)(x+a)}.$$

167. We may now enunciate the rule for the addition or subtraction of fractions.

**Rule II.** *To add or subtract fractions: reduce them to the lowest common denominator; add or subtract the numerators, and retain the common denominator.*

*Example* 1. Find the value of $\frac{2x+a}{3a}+\frac{5x-4a}{9a}$.

The lowest common denominator is $9a$.

$$\text{Therefore the expression} = \frac{3(2x+a)+5x-4a}{9a}$$

$$= \frac{6x+3a+5x-4a}{9a} = \frac{11x-a}{9a}.$$

*Example* 2. Find the value of $\frac{x-2y}{xy}+\frac{3y-a}{ay}+\frac{3x-2a}{ax}$.

The lowest common denominator is $axy$.

$$\text{Thus the expression} = \frac{a(x-2y)+x(3y-a)-y(3x-2a)}{axy}$$

$$= \frac{ax-2ay+3xy-ax-3xy+2ay}{axy}$$

$$=0,$$

since the terms in the numerator destroy each other.

**Note.** To ensure accuracy the beginner is recommended to use brackets as in the first line of work above.

## EXAMPLES XXI. a.

Find the value of

1. $\frac{x-1}{2}+\frac{x+3}{5}+\frac{x+7}{10}$.

2. $\frac{2x-1}{3}+\frac{x-5}{6}+\frac{x-4}{4}$.

3. $\frac{5x-1}{8}-\frac{3x-2}{7}+\frac{x-5}{4}$.

4. $\frac{2x-3}{9}-\frac{x+2}{6}+\frac{5x+8}{12}$.

5. $\frac{x-7}{15}+\frac{x-9}{25}-\frac{x+3}{45}$.

6. $\frac{2x+5}{x}-\frac{x+3}{2x}-\frac{27}{8x^2}$.

7. $\frac{a-b}{ab}+\frac{b-c}{bc}+\frac{c-a}{ca}$.

8. $\frac{a-2b}{2a}-\frac{a-5b}{4a}+\frac{a+7b}{8a}$.

9. $\frac{b+c}{2a}+\frac{c+a}{4b}-\frac{a-b}{3c}$.

10. $\frac{a-x}{x}+\frac{a+x}{a}-\frac{a^2-x^2}{2ax}$.

11. $\frac{x+2}{17x}-\frac{x-5}{34x}+\frac{x+2}{51x}$.

12. $\frac{2a^2-b^2}{a^2}-\frac{b^2-c^2}{b^2}-\frac{c^2-a^2}{c^2}$.

13. $\frac{x-3}{5x}+\frac{x^2-9}{10x^2}-\frac{8-x^3}{15x^3}$.

14. $\frac{2}{xy}-\frac{3y^2-x^2}{xy^3}+\frac{xy+y^2}{x^2y^2}$.

15. $\frac{2x-3y}{xy}+\frac{3x-2z}{xz}+\frac{5}{x}$.

16. $\frac{a^2-bc}{bc}-\frac{ac-b^2}{ac}-\frac{ab-c^2}{ab}$.

*Example* 3. Simplify $\frac{2x-3a}{x-2a}-\frac{2x-a}{x-a}$.

The lowest common denominator is $(x-2a)(x-a)$.

Hence, multiplying the numerators by $x-a$ and $x-2a$ respectively, we have

$$\text{the expression}=\frac{(2x-3a)(x-a)-(2x-a)(x-2a)}{(x-2a)(x-a)}$$

$$=\frac{2x^2-5ax+3a^2-(2x^2-5ax+2a^2)}{(x-2a)(x-a)}$$

$$=\frac{2x^2-5ax+3a^2-2x^2+5ax-2a^2}{(x-2a)(x-a)}$$

$$=\frac{a^2}{(x-2a)(x-a)}.$$

NOTE. In finding the value of such an expression as

$$-(2x-a)(x-2a),$$

the beginner should first express the product in brackets, and then remove the brackets, as we have done. After a little practice he will be able to take both steps together.

The work will sometimes be shortened by first reducing the fractions to their lowest terms.

*Example* 4. Simplify $\dfrac{x^2+5xy-4y^2}{x^2-16y^2}-\dfrac{2xy}{2x^2+8xy}$.

$$\begin{aligned}\text{The expression}&=\frac{x^2+5xy-4y^2}{x^2-16y^2}-\frac{y}{x+4y}\\&=\frac{x^2+5xy-4y^2-y(x-4y)}{x^2-16y^2}\\&=\frac{x^2+5xy-4y^2-xy+4y^2}{x^2-16y^2}\\&=\frac{x^2+4xy}{x^2-16y^2}=\frac{x}{x-4y}.\end{aligned}$$

## EXAMPLES XXI. b.

Find the value of

1. $\dfrac{1}{x+2}+\dfrac{1}{x+3}$.
2. $\dfrac{2}{x+3}-\dfrac{1}{x+4}$.
3. $\dfrac{1}{x-5}-\dfrac{1}{x-4}$.
4. $\dfrac{3}{x-6}-\dfrac{1}{x+2}$.
5. $\dfrac{a}{x+a}-\dfrac{b}{x+b}$.
6. $\dfrac{a}{x-a}+\dfrac{b}{x-b}$.
7. $\dfrac{x+3}{x+4}-\dfrac{x+1}{x+2}$.
8. $\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}$.
9. $\dfrac{x+2}{x-2}-\dfrac{x-2}{x+2}$.
10. $\dfrac{x-4}{x-2}-\dfrac{x-7}{x-5}$.
11. $\dfrac{a}{x-a}-\dfrac{a^2}{x^2-a^2}$.
12. $\dfrac{3}{x-3}+\dfrac{2x}{x^2-9}$.
13. $\dfrac{1}{2x-3y}-\dfrac{x+y}{4x^2-9y^2}$.
14. $\dfrac{x+a}{x-2a}-\dfrac{x^2+2a^2}{x^2-4a^2}$.
15. $\dfrac{4a^2+b^2}{4a^2-b^2}-\dfrac{2a-b}{2a+b}$.
16. $\dfrac{2x^2}{x^2-y^2}-\dfrac{2x}{x+y}$.
17. $\dfrac{x}{1-x^2}-\dfrac{x}{1+x^2}$.
18. $\dfrac{1}{x(x-y)}+\dfrac{1}{y(x+y)}$.
19. $\dfrac{xy}{25x^2-y^2}+\dfrac{x}{5x+y}$.
20. $\dfrac{y}{x(x^2-y^2)}+\dfrac{x}{y(x^2+y^2)}$.
21. $\dfrac{x^2-4a^2}{x^2-2ax}-\dfrac{x+4a}{x+2a}$.
22. $\dfrac{x^2+xy+y^2}{x+y}+\dfrac{x^2-xy+y^2}{x-y}$.
23. $\dfrac{1}{a-2x}-\dfrac{(a+2x)^2}{a^3-8x^3}$.
24. $\dfrac{a^3+b^3}{a^2-ab+b^2}-\dfrac{a^3-b^3}{a^2+ab+b^2}$.

168. Some modification of the foregoing general methods may sometimes be used with advantage. The most useful artifices are explained in the examples which follow, but no general rules can be given which will apply to all cases.

*Example* 1. Simplify $\frac{a+3}{a-4}-\frac{a+4}{a-3}-\frac{8}{a^2-16}$.

Taking the first two fractions together, we have

$$\text{the expression} = \frac{a^2-9-(a^2-16)}{(a-4)(a-3)}-\frac{8}{a^2-16}$$

$$=\frac{7}{(a-4)(a-3)}-\frac{8}{(a+4)(a-4)}$$

$$=\frac{7(a+4)-8(a-3)}{(a+4)(a-4)(a-3)}$$

$$=\frac{52-a}{(a+4)(a-4)(a-3)}.$$

*Example* 2. Simplify $\frac{1}{2x^2+x-1}+\frac{1}{3x^2+4x+1}$.

$$\text{The expression} = \frac{1}{(2x-1)(x+1)}+\frac{1}{(3x+1)(x+1)}$$

$$=\frac{3x+1+2x-1}{(2x-1)(x+1)(3x+1)}$$

$$=\frac{5x}{(2x-1)(x+1)(3x+1)}.$$

*Example* 3. Simplify $\frac{1}{a-x}-\frac{1}{a+x}-\frac{2x}{a^2+x^2}-\frac{4x^3}{a^4+x^4}$.

Here it should be evident that the first two denominators give L. C. M. $a^2-x^2$, which readily combines with $a^2+x^2$ to give L. C. M. $a^4-x^4$, which again combines with $a^4+x^4$ to give L. C. M. $a^8-x^8$. Hence it will be convenient to proceed as follows:

$$\text{The expression} = \frac{a+x-(a-x)}{a^2-x^2}-\dots\dots\dots$$

$$=\frac{2x}{a^2-x^2}-\frac{2x}{a^2+x^2}-\dots\dots$$

$$=\frac{4x^3}{a^4-x^4}-\frac{4x^3}{a^4+x^4}$$

$$=\frac{8x^7}{a^8-x^8}.$$

## EXAMPLES XXI. c.

Find the value of

1. $\frac{1}{x+y}-\frac{1}{x-y}+\frac{2x}{x^2-y^2}$.    2. $\frac{1}{2x+y}+\frac{1}{2x-y}-\frac{3x}{4x^2-y^2}$.

3. $\frac{5}{1+2x}-\frac{3x}{1-2x}-\frac{4-13x}{1-4x^2}$.

4. $\frac{2a}{2a+3b}+\frac{3b}{2a-3b}-\frac{8b^2}{4a^2-9b^2}$.

5. $\frac{10}{9-a^2}-\frac{2}{3+a}-\frac{1}{3-a}$.    6. $\frac{5x}{6(x^2-1)}-\frac{1}{2(x-1)}+\frac{1}{3(x+1)}$.

7. $\frac{1}{2(a-b)}-\frac{1}{2(a+b)}-\frac{b}{a^2-b^2}$.

8. $\frac{2a}{2a-3}-\frac{5}{6a+9}-\frac{4(3a+2)}{3(4a^2-9)}$.

9. $\frac{3}{x-2}+\frac{2}{3x+6}+\frac{5x}{x^2-4}$.    10. $\frac{x}{x^3+y^3}-\frac{y}{x^3-y^3}+\frac{x^3y+xy^3}{x^6-y^6}$.

11. $\frac{1}{x^2-9x+20}+\frac{1}{x^2-11x+30}$.

12. $\frac{1}{x^2-7x+12}-\frac{1}{x^2-5x+6}$.

13. $\frac{1}{2x^2-x-1}-\frac{1}{2x^2+x-3}$.    14. $\frac{1}{2x^2-x-1}-\frac{3}{6x^2-x-2}$.

15. $\frac{4}{4-7a-2a^2}-\frac{3}{3-a-10a^2}$.

16. $\frac{5}{5+x-18x^2}-\frac{2}{2+5x+2x^2}$.

17. $\frac{1}{x+1}-\frac{1}{(x+1)(x+2)}+\frac{1}{(x+1)(x+2)(x+3)}$.

18. $\frac{5x}{2(x+1)(x-3)}-\frac{15(x-1)}{16(x-3)(x-2)}-\frac{9(x+3)}{16(x+1)(x-2)}$.

19. $\frac{a+3b}{4(a+b)(a+2b)}+\frac{a+2b}{(a+b)(a+3b)}-\frac{a+b}{4(a+2b)(a+3b)}$.

20. $\frac{2}{x^2-3x+2}+\frac{2}{x^2-x-2}-\frac{1}{x^2-1}$.

21. $\frac{x}{x^2+5x+6}+\frac{15}{x^2+9x+14}-\frac{12}{x^2+10x+21}$.

22. $\frac{3}{x^2-1}+\frac{4}{2x+1}+\frac{4x+2}{2x^2+3x+1}$.

23. $\frac{5(2x-3)}{11(6x^2+x-1)}+\frac{7x}{6x^2+7x-3}-\frac{12(3x+1)}{11(4x^2+8x+3)}$.

24. $\frac{x-3}{x+2}-\frac{x-2}{x+3}+\frac{1}{x-1}$. 25. $\frac{x-3}{x-4}-\frac{x+4}{x+3}-\frac{5}{x^2-16}$.

26. $\frac{1+2a}{1-2a}-\frac{1-2a}{1+2a}-\frac{8a}{(1-2a)^2}$.

27. $\frac{24x}{9-12x+4x^2}-\frac{3+2x}{3-2x}+\frac{3-2x}{3+2x}$.

28. $\frac{1}{3-x}-\frac{1}{3+x}-\frac{2x}{9+x^2}$. 29. $\frac{1}{2a+3}+\frac{1}{2a-3}-\frac{4a}{4a^2+9}$.

30. $\frac{1}{4(1+x)}+\frac{1}{4(1-x)}+\frac{1}{2(1+x^2)}$.

31. $\frac{3}{8(a-x)}+\frac{1}{8(a+x)}-\frac{a-x}{4(a^2+x^2)}$.

32. $\frac{2x}{4+x^2}+\frac{1}{2-x}-\frac{1}{2+x}$. 33. $\frac{5}{3-6x}-\frac{5}{3+6x}-\frac{x}{2+8x^2}$.

34. $\frac{1}{2a-8x}-\frac{a}{3a^2+48x^2}+\frac{1}{2a+8x}$.

35. $\frac{1}{6a^2+54}+\frac{1}{3a-9}-\frac{a}{3a^2-27}$.

36. $\frac{1}{8-8x}-\frac{1}{8+8x}+\frac{x}{4+4x^2}-\frac{x}{2+2x^4}$.

37. $\frac{1}{6a-18}-\frac{1}{6a+18}-\frac{1}{a^2+9}+\frac{18}{a^4+81}$.

169. We have not yet attempted to explain in what sense a fraction may be regarded when its numerator and denominator are not both positive. The definition of a fraction, Art. 150, is no longer applicable; but it is not difficult to give an interpretation which is intelligible, and in accordance with the principles already laid down.

Since $\frac{a}{b}\times b=a$, the fraction $\frac{a}{b}$ is the quantity which must be multiplied by $b$ in order to obtain $a$; so that *the fraction* $\frac{\mathrm{a}}{\mathrm{b}}$ *may be regarded as the quotient resulting from the division of* a *by* b.

We may here remind the student that in division, when divisor and dividend are not both positive, we perform the division just as if they were positive, and afterwards prefix to the quotient the appropriate sign given by *the rule of signs.*

170. To find a meaning for the symbol $\frac{-a}{-b}$, we define it as the quotient resulting from the division of $-a$ by $-b$; and this is obtained by dividing $a$ by $b$, and, by the rule of signs, prefixing $+$.

Therefore
$$\frac{-a}{-b} = +\frac{a}{b} = \frac{a}{b} \text{ ........................... } (1).$$

Again, $\frac{-a}{b}$ is the quotient resulting from the division of $-a$ by $b$; and this is obtained by dividing $a$ by $b$, and, by the rule of signs, prefixing $-$.

Therefore
$$\frac{-a}{b} = -\frac{a}{b} \text{ ................................. } (2).$$

Likewise $\frac{a}{-b}$ is the quotient resulting from the division of $a$ by $-b$; and this is obtained by dividing $a$ by $b$, and, by the rule of signs, prefixing $-$

Therefore
$$\frac{a}{-b} = -\frac{a}{b} \text{ ................................. } (3).$$

These results may be enunciated as follows:

(1) *If the signs of both numerator and denominator of a fraction be changed, the sign of the whole fraction will be unchanged.*

(2) *If the sign of the numerator alone be changed, the sign of the whole fraction will be changed.*

(3) *If the sign of the denominator alone be changed, the sign of the whole fraction will be changed.*

The principles here involved are so useful in certain cases of reduction of fractions that we quote them in another form, which will sometimes be found more easy of application.

1. *We may change the sign of every term in the numerator and denominator of a fraction without altering its value.*

2. *We may change the sign of a fraction by simply changing the sign of every term either in the numerator or denominator.*

*Example* 1. $$\frac{b-a}{y-x}=\frac{-b+a}{-y+x}=\frac{a-b}{x-y}.$$

*Example* 2. $$\frac{x-x^2}{2y}=-\frac{-x+x^2}{2y}=-\frac{x^2-x}{2y}.$$

*Example* 3. $$\frac{3x}{4-x^2}=-\frac{3x}{-4+x^2}=-\frac{3x}{x^2-4}.$$

The intermediate step may usually be omitted.

*Example* 4. Simplify $\dfrac{a}{x+a}+\dfrac{2x}{x-a}+\dfrac{a(3x-a)}{a^2-x^2}$.

Here it is evident that the lowest common denominator of the first two fractions is $x^2-a^2$, therefore it will be convenient to alter the sign of the denominator in the third fraction.

$$\begin{aligned}\text{Thus the expression}&=\frac{a}{x+a}+\frac{2x}{x-a}-\frac{a(3x-a)}{x^2-a^2}\\&=\frac{a(x-a)+2x(x+a)-a(3x-a)}{x^2-a^2}\\&=\frac{ax-a^2+2x^2+2ax-3ax+a^2}{x^2-a^2}\\&=\frac{2x^2}{x^2-a^2}.\end{aligned}$$

*Example* 5. Simplify $\dfrac{5}{3x-3}+\dfrac{3x-1}{1-x^2}+\dfrac{1}{2x+2}$.

$$\begin{aligned}\text{The expression}&=\frac{5}{3(x-1)}-\frac{3x-1}{x^2-1}+\frac{1}{2(x+1)}\\&=\frac{10(x+1)-6(3x-1)+3(x-1)}{6(x^2-1)}\\&=\frac{10x+10-18x+6+3x-3}{6(x^2-1)}\\&=\frac{13-5x}{6(x^2-1)}.\end{aligned}$$

## EXAMPLES XXI. d.

Simplify

1. $\dfrac{1}{4x-4}-\dfrac{1}{5x+5}+\dfrac{1}{1-x^2}$. 2. $\dfrac{3}{1+a}-\dfrac{2}{1-a}-\dfrac{5a}{a^2-1}$.

3. $\dfrac{x-2a}{x+a}+\dfrac{2(a^2-4ax)}{a^2-x^2}-\dfrac{3a}{x-a}$.

**4.** $\dfrac{x-a}{x+a}+\dfrac{a^2+3ax}{a^2-x^2}+\dfrac{x+a}{x-a}.$ **5.** $\dfrac{1}{2x+1}+\dfrac{1}{2x-1}+\dfrac{4x}{1-4x^2}.$

**6.** $\dfrac{3x}{1-x^2}-\dfrac{2}{x-1}-\dfrac{2}{x+1}.$

**7.** $\dfrac{2-5x}{x+3}-\dfrac{3+x}{3-x}+\dfrac{2x(2x-11)}{x^2-9}.$

**8.** $\dfrac{3-2x}{2x+3}-\dfrac{2x+3}{3-2x}+\dfrac{12}{4x^2-9}.$ **9.** $\dfrac{5}{2b+2}-\dfrac{3}{4b-4}+\dfrac{11}{6-6b^2}.$

**10.** $\dfrac{1}{6a+6}+\dfrac{1}{6-6a}-\dfrac{1}{3a^2-3}.$ **11.** $\dfrac{y^2}{x^3-y^3}+\dfrac{x^3y^2}{y^6-x^6}.$

**12.** $\dfrac{x^2-y^2}{xy}-\dfrac{xy-y^2}{xy-x^2}.$ **13.** $\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{x}{x+y}+\dfrac{y}{y-x}.$

**14.** $\dfrac{x^2+2x+4}{x+2}-\dfrac{x^2-2x+4}{2-x}.$

**15.** $\dfrac{1}{2a+5b}+\dfrac{3a}{25b^2-4a^2}+\dfrac{1}{2a-5b}.$

**16.** $\dfrac{2b-a}{x-b}-\dfrac{3x(a-b)}{b^2-x^2}+\dfrac{b-2a}{b+x}.$

**17.** $\dfrac{ax^2+b}{2x-1}+\dfrac{2(bx+ax^2)}{1-4x^2}-\dfrac{ax^2-b}{2x+1}.$

**18.** $\dfrac{a+c}{(a-b)(x-a)}+\dfrac{b+c}{(b-a)(x-b)}.$

**19.** $\dfrac{a-c}{(a-b)(x-a)}-\dfrac{b-c}{(b-a)(b-x)}.$

**20.** $\dfrac{2a+y}{(x-a)(a-b)}+\dfrac{a+b+y}{(x-b)(b-a)}-\dfrac{x+y-a}{(x-a)(x-b)}.$

**21.** $\dfrac{1}{(a^2-b^2)(x^2+b^2)}+\dfrac{1}{(b^2-a^2)(x^2+a^2)}-\dfrac{1}{(x^2+a^2)(x^2+b^2)}.$

**22.** $\dfrac{1}{x+a}+\dfrac{4a}{x^2-a^2}+\dfrac{1}{a-x}-\dfrac{2a}{x^2+a^2}.$

**23.** $\dfrac{3}{x+a}-\dfrac{1}{x+3a}+\dfrac{3}{a-x}+\dfrac{1}{x-3a}.$

**24.** $\dfrac{1}{4a^3(a+x)}-\dfrac{1}{4a^3(x-a)}+\dfrac{1}{2a^2(a^2+x^2)}-\dfrac{a^4}{a^8-x^8}.$

**25.** $\dfrac{x}{x^2-y^2}-\dfrac{y}{x^2+y^2}+\dfrac{x^3+y^3}{y^4-x^4}+\dfrac{xy}{(x+y)(x^2+y^2)}.$

**26.** $\dfrac{b}{a(a^2-b^2)}+\dfrac{a}{b(a^2+b^2)}+\dfrac{a^4+b^4}{ab(b^4-a^4)}-\dfrac{a^6}{b^8-a^8}.$

*171. Consider the expression

$$\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}.$$

Here in finding the L.C.M. of the denominators it must be observed that there are not *six* different compound factors to be considered; for three of them differ from the other three only in sign.

Thus
$$(a-c)=-(c-a),$$
$$(b-a)=-(a-b),$$
$$(c-b)=-(b-c).$$

Hence, replacing the second factor in each denominator by its equivalent, we may write the expression in the form

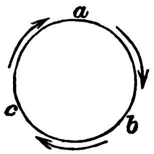
$$-\frac{1}{(a-b)(c-a)}-\frac{1}{(b-c)(a-b)}-\frac{1}{(c-a)(b-c)} \text{.........(1).}$$

Now the L.C.M. is $(b-c)(c-a)(a-b)$;

and the expression $=\dfrac{-(b-c)-(c-a)-(a-b)}{(b-c)(c-a)(a-b)}$

$$=\frac{-b+c-c+a-a+b}{(b-c)(c-a)(a-b)}$$
$$=0.$$

*172. There is a peculiarity in the arrangement of this example which it is desirable to notice. In the expression (1) the letters occur in what is known as **Cyclic Order**; that is, $b$ follows $a$, $a$ follows $c$. Thus if $a$, $b$, $c$ are arranged round the circumference of a circle, as in the annexed diagram, if we start from any letter and move round in the direction of the arrows, the other letters follow in cyclic order, namely *abc*, *bca*, *cab*.



The observance of this principle is especially important in a large class of examples in which the differences of three letters are involved. Thus we are observing cyclic order when we write $b-c$, $c-a$, $a-b$; whereas we are violating cyclic order by the use of arrangements such as $b-c$, $a-c$, $a-b$, or $a-c$, $b-a$, $b-c$. It will always be found that the work is rendered shorter and easier by following cyclic order from the beginning, and adhering to it throughout the question.

In the present chapter we shall confine our attention to a few of the simpler cases, resuming the subject in Chapter XXIX.

## *EXAMPLES XXI. e.

Find the value of

1. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}$.

2. $\dfrac{b}{(a-b)(a-c)}+\dfrac{c}{(b-c)(b-a)}+\dfrac{a}{(c-a)(c-b)}$.

3. $\dfrac{z}{(x-y)(x-z)}+\dfrac{x}{(y-z)(y-x)}+\dfrac{y}{(z-x)(z-y)}$.

4. $\dfrac{y+z}{(x-y)(x-z)}+\dfrac{z+x}{(y-z)(y-x)}+\dfrac{x+y}{(z-x)(z-y)}$.

5. $\dfrac{b-c}{(a-b)(a-c)}+\dfrac{c-a}{(b-c)(b-a)}+\dfrac{a-b}{(c-a)(c-b)}$.

6. $\dfrac{x^2yz}{(x-y)(x-z)}+\dfrac{y^2zx}{(y-z)(y-x)}+\dfrac{z^2xy}{(z-x)(z-y)}$.

7. $\dfrac{1+a}{(a-b)(a-c)}+\dfrac{1+b}{(b-c)(b-a)}+\dfrac{1+c}{(c-a)(c-b)}$.

8. $\dfrac{p-a}{(p-q)(p-r)}+\dfrac{q-a}{(q-r)(q-p)}+\dfrac{r-a}{(r-p)(r-q)}$.

9. $\dfrac{p+q-r}{(p-q)(p-r)}+\dfrac{q+r-p}{(q-r)(q-p)}+\dfrac{r+p-q}{(r-p)(r-q)}$.

10. $\dfrac{a^2}{(a^2-b^2)(a^2-c^2)}+\dfrac{b^2}{(b^2-c^2)(b^2-a^2)}+\dfrac{c^2}{(c^2-a^2)(c^2-b^2)}$.

11. $\dfrac{x+y}{(p-q)(p-r)}+\dfrac{x+y}{(q-r)(q-p)}+\dfrac{x+y}{(r-p)(r-q)}$.

12. $\dfrac{q+r}{(x-y)(x-z)}+\dfrac{r+p}{(y-z)(y-x)}+\dfrac{p+q}{(z-x)(z-y)}$.

# CHAPTER XXII.

## MISCELLANEOUS FRACTIONS.

173. WE now propose to consider some miscellaneous questions involving fractions of a more complicated kind than those already discussed.

In the previous Chapters on Fractions, the numerator and denominator have been regarded as integers; but cases frequently occur in which the numerator or denominator of a fraction is itself fractional.

174. DEFINITION. A fraction whose numerator and denominator are whole numbers is called a **Simple Fraction.**

A fraction of which the numerator or denominator is itself a fraction is called a **Complex Fraction.**

Thus $\dfrac{\frac{a}{b}}{c}$, $\dfrac{\frac{a}{b}}{x}$, $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ are Complex Fractions.

In the last of these types, the outside quantities, $a$ and $d$, are sometimes referred to as the *extremes*, while the two middle quantities, $b$ and $c$, are called the *means*.

*175. DEFINITION. If we divide the unit into any number of equal parts, each part is called a **Sub-unit.**

*176. Consider the fraction $\dfrac{\frac{a}{b}}{\frac{c}{d}}$.

If this is to obey the definition, it must represent $\frac{a}{b}$ parts, $\frac{c}{d}$ of which make up the unit.

Divide the unit into $bc$ sub-units.

Thus $\frac{c}{d}$ parts make up the unit, and are therefore equivalent to $bc$ sub-units.

Hence $c$ parts are equivalent to $bcd$ sub-units;

$\therefore$ 1 part is .................. $bd$ sub-units;

$\therefore$ $\frac{a}{b}$ parts is .................. $ad$ sub-units;

that is the fraction $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ represents $ad$ sub-units, $bc$ of which make up the unit; but this, by definition, is the fraction $\frac{ad}{bc}$.

$$\therefore \qquad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}.$$

177. From the preceding Article we deduce an easy method of writing down the simplified form of a complex fraction.

*Multiply the extremes for a new numerator, and the means for a new denominator.*

*Example.* $$\frac{\frac{a+x}{b}}{\frac{a^2-x^2}{ab}} = \frac{ab\,(a+x)}{b\,(a^2-x^2)} = \frac{a}{a-x},$$

by cancelling common factors in numerator and denominator.

178. We have proved that

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}; \qquad \text{[Art. 158.]}$$

and $$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}; \qquad \text{[Art. 176.]}$$

$$\therefore \qquad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d}.$$

We have previously shewn that, when the numerator and denominator are integers, a fraction may be regarded as representing the quotient of the numerator by the denominator. We now see that a complex fraction may be regarded in the same sense.

179. The student should especially notice the following cases, and should be able to write down the results readily.

$$\frac{1}{\frac{a}{b}} = 1 \div \frac{a}{b} = 1 \times \frac{b}{a} = \frac{b}{a},$$

$$\frac{a}{\frac{1}{b}} = a \div \frac{1}{b} = a \times b = ab.$$

$$\frac{\frac{1}{a}}{\frac{1}{b}} = \frac{1}{a} \div \frac{1}{b} = \frac{1}{a} \times \frac{b}{1} = \frac{b}{a}.$$

180. We now proceed to shew how complex fractions can be reduced by the rules already given.

*Example* 1.
$$\begin{aligned}\frac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}} &= \left(\frac{a}{b}+\frac{c}{d}\right) \div \left(\frac{a}{b}-\frac{c}{d}\right)\\ &= \frac{ad+bc}{bd} \div \frac{ad-bc}{bd}\\ &= \frac{ad+bc}{bd} \times \frac{bd}{ad-bc}\\ &= \frac{ad+bc}{ad-bc}.\end{aligned}$$

*Example* 2.
$$\begin{aligned}\frac{x+\frac{a^2}{x}}{x-\frac{a^4}{x^3}} &= \left(x+\frac{a^2}{x}\right) \div \left(x-\frac{a^4}{x^3}\right)\\ &= \frac{x^2+a^2}{x} \div \frac{x^4-a^4}{x^3}\\ &= \frac{x^2+a^2}{x} \times \frac{x^3}{x^4-a^4}\\ &= \frac{x^2}{x^2-a^2}.\end{aligned}$$

*Example* 3. Simplify $\dfrac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$.

The numerator $=\dfrac{(a^2+b^2)^2-(a^2-b^2)^2}{(a^2+b^2)(a^2-b^2)}$

$$=\frac{4a^2b^2}{(a^2+b^2)(a^2-b^2)}.$$

Similarly the denominator $=\dfrac{4ab}{(a+b)(a-b)}$.

Hence the fraction $=\dfrac{4a^2b^2}{(a^2+b^2)(a^2-b^2)}\div\dfrac{4ab}{(a+b)(a-b)}$

$$=\frac{4a^2b^2}{(a^2+b^2)(a^2-b^2)}\times\frac{(a+b)(a-b)}{4ab}$$

$$=\frac{ab}{a^2+b^2}.$$

**Note.** To ensure accuracy and neatness, when the numerator and denominator are somewhat complicated, the beginner is advised to simplify each separately as in the above example.

In the case of **Continued Fractions**, we begin from the lowest fraction, and simplify step by step.

*Example* 4. Simplify $\dfrac{9x^2-64}{x-1-\dfrac{1}{1-\dfrac{x}{4+x}}}$.

The expression $=\dfrac{9x^2-64}{x-1-\dfrac{1}{\dfrac{4+x-x}{4+x}}}$

$$=\frac{9x^2-64}{x-1-\dfrac{4+x}{4}}$$

$$=\frac{9x^2-64}{\dfrac{4x-4-(4+x)}{4}}=\frac{9x^2-64}{\dfrac{3x-8}{4}}$$

$$=\frac{4(9x^2-64)}{3x-8}=4(3x+8).$$

## EXAMPLES XXII. a.

Find the value of

1. $\dfrac{\dfrac{m}{n}-\dfrac{l}{m}}{\dfrac{a}{m}-\dfrac{b}{n}}$.  2. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x}-\dfrac{1}{y}}$.  3. $\dfrac{a+\dfrac{b}{d}}{x-\dfrac{y}{d}}$.  4. $\dfrac{1+\dfrac{c}{x}}{\dfrac{b}{x}-1}$.

5. $\dfrac{2+\dfrac{3a}{4b}}{a+\dfrac{8b}{3}}$.  6. $\dfrac{3a+\dfrac{7b}{8c}}{3c+\dfrac{7b}{8a}}$.  7. $\dfrac{1-\dfrac{y^2}{x^2}}{1+\dfrac{y^2}{x^2}}$.  8. $\dfrac{1}{a+\dfrac{b}{c}}$.

9. $\dfrac{a}{b+\dfrac{c}{d}}$.  10. $\dfrac{x}{x-\dfrac{m}{n}}$.  11. $\dfrac{\dfrac{a}{b}+\dfrac{c}{d}}{\dfrac{m}{n}+\dfrac{k}{p}}$.  12. $\dfrac{x-\dfrac{1}{x}}{1+\dfrac{1}{x}}$.

13. $\dfrac{x+5+\dfrac{6}{x}}{1+\dfrac{6}{x}+\dfrac{8}{x^2}}$.  14. $\dfrac{\dfrac{1}{x}-\dfrac{2}{x^2}-\dfrac{3}{x^3}}{\dfrac{9}{x}-x}$.  15. $\dfrac{2x^2-x-6}{\dfrac{4}{x^2}-1}$.

16. $\dfrac{2}{1-x^2}\div\left(\dfrac{1}{1-x}-\dfrac{1}{1+x}\right)$.  17. $\left(\dfrac{a^3-b^3}{a-b}-\dfrac{a^3+b^3}{a+b}\right)\div\dfrac{4ab}{a^2-b^2}$.

18. $\left(\dfrac{a^2-ax+x^2}{a-x}-\dfrac{a^2+ax+x^2}{a+x}\right)\div\dfrac{x^3}{a^2-x^2}$.

19. $\left(y+\dfrac{xy}{y-x}\right)\left(y-\dfrac{xy}{x+y}\right)\times\dfrac{y^2-x^2}{y^2+x^2}$.

20. $\left(\dfrac{x}{1+x}+\dfrac{1-x}{x}\right)\div\left(\dfrac{x}{1+x}-\dfrac{1-x}{x}\right)$.

21. $\dfrac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{1-\dfrac{a^2+b^2}{(a+b)^2}}$.  22. $\dfrac{\dfrac{a}{x^2}+\dfrac{x}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{ax}+\dfrac{1}{x^2}}$.

23. $\dfrac{\dfrac{1}{3x-2}-\dfrac{1}{3x+2}}{\dfrac{1}{9-\dfrac{4}{x^2}}}$.  24. $1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}$.

25. $\dfrac{1}{a-\dfrac{a^2-1}{a+\dfrac{1}{a-1}}}$.

26. $\dfrac{1}{4x+\dfrac{4x}{1+\dfrac{2(x+y)}{6-x}}}$.

27. $\dfrac{a}{x+\dfrac{m}{y+\dfrac{n}{z}}}$.

28. $\dfrac{1}{1-\dfrac{1+x}{x-\dfrac{1}{x}}}$.

29. $\dfrac{x-2}{x-2-\dfrac{x}{x-\dfrac{x-1}{x-2}}}$.

30. $\dfrac{1}{x-\dfrac{1}{x+\dfrac{1}{x}}}-\dfrac{1}{x+\dfrac{1}{x-\dfrac{1}{x}}}$.

181. Sometimes it is convenient to express a single fraction as a group of fractions.

*Example.*
$$\frac{5x^2y-10xy^2+15y^3}{10x^2y^2}=\frac{5x^2y}{10x^2y^2}-\frac{10xy^2}{10x^2y^2}+\frac{15y^3}{10x^2y^2}$$
$$=\frac{1}{2y}-\frac{1}{x}+\frac{3y}{2x^2}.$$

182. Since a fraction represents the quotient of the numerator by the denominator, we may often express a fraction in an equivalent form, partly integral and partly fractional.

*Example* 1. $\dfrac{x+7}{x+2}=\dfrac{(x+2)+5}{x+2}=1+\dfrac{5}{x+2}$.

*Example* 2. $\dfrac{3x-2}{x+5}=\dfrac{3(x+5)-15-2}{x+5}=\dfrac{3(x+5)-17}{x+5}=3-\dfrac{17}{x+5}$.

In some cases actual division may be advisable.

*Example* 3. Shew that $\dfrac{2x^2-7x-1}{x-3}=2x-1-\dfrac{4}{x-3}$.

By division
$$\begin{array}{l}
x-3\,)\,2x^2-7x-1\,(\,2x-1 \\
\qquad\quad\ \underline{2x^2-6x\phantom{-1}} \\
\qquad\qquad\ \ -x-1 \\
\qquad\qquad\ \ \underline{-x+3} \\
\qquad\qquad\qquad -4.
\end{array}$$

Thus the quotient is $2x-1$, and the remainder $-4$.

Therefore $\dfrac{2x^2-7x-1}{x-3}=2x-1-\dfrac{4}{x-3}$.

183. If the numerator be of lower dimensions than the denominator, we may still perform the division, and express the result in a form which is partly integral and partly fractional.

*Example.* Prove that $\dfrac{2x}{1+3x^2}=2x-6x^3+18x^5-\dfrac{54x^7}{1+3x^2}$.

By division

$$\begin{array}{rl}
1+3x^2)\,2x & (2x-6x^3+18x^5\\
2x+6x^3 & \\
\hline
-6x^3 & \\
-6x^3-18x^5 & \\
\hline
18x^5 & \\
18x^5+54x^7 & \\
\hline
-54x^7 &
\end{array}$$

whence the result follows.

Here the division may be carried on to any number of terms in the quotient, and we can stop at any term we please by taking for our remainder the fraction whose numerator is the remainder last found, and whose denominator is the divisor.

Thus, if we carried on the quotient to four terms, we should have

$$\frac{2x}{1+3x^2}=2x-6x^3+18x^5-54x^7+\frac{162x^9}{1+3x^2}.$$

The terms in the quotient may be fractional; thus if $x^2$ is divided by $x^3-a^3$, the first four terms of the quotient are $\dfrac{1}{x}+\dfrac{a^3}{x^4}+\dfrac{a^6}{x^7}+\dfrac{a^9}{x^{10}}$, and the remainder is $\dfrac{a^{12}}{x^{10}}$.

184. Miscellaneous examples in multiplication and division occur which can be dealt with by the preceding rules for the reduction of fractions.

*Example.* Multiply $x+2a-\dfrac{a^2}{2x+3a}$ by $2x-a-\dfrac{2a^2}{x+a}$.

$$\begin{aligned}
\text{The product}&=\left(x+2a-\frac{a^2}{2x+3a}\right)\times\left(2x-a-\frac{2a^2}{x+a}\right)\\
&=\frac{2x^2+7ax+6a^2-a^2}{2x+3a}\times\frac{2x^2+ax-a^2-2a^2}{x+a}\\
&=\frac{2x^2+7ax+5a^2}{2x+3a}\times\frac{2x^2+ax-3a^2}{x+a}\\
&=\frac{(2x+5a)\,(x+a)}{2x+3a}\times\frac{(2x+3a)\,(x-a)}{x+a}\\
&=(2x+5a)\,(x-a).
\end{aligned}$$

## EXAMPLES XXII. b.

Express each of the following fractions as a group of simple fractions in lowest terms:

1. $\dfrac{3x^2y+xy^2-y^3}{9xy}$.

2. $\dfrac{3a^3x-4a^2x^2+6ax^3}{12ax}$.

3. $\dfrac{a^3-3a^2b+3ab^2+b^3}{2ab}$.

4. $\dfrac{a+b+c}{abc}$.

5. $\dfrac{bc+ca+ab}{abc}$.

6. $\dfrac{a^3bc-3ab^3c+2abc}{6abc}$.

Perform the following divisions, giving the remainder after four terms in the quotient:

7. $x\div(1+x)$.

8. $a\div(a-b)$.

9. $(1+x)\div(1-x)$.

10. $1\div(1-x+x^2)$.

11. $x^2\div(x+3)$.

12. $1\div(1-x)^2$.

13. Shew that $\dfrac{a^3-b^3}{(a-b)^2}=a+2b+\dfrac{3b^2}{a-b}$.

14. Shew that $x^2-xy+y^2-\dfrac{2y^3}{x+y}=\dfrac{x^3-y^3}{x+y}$.

15. Shew that $\dfrac{60x^3-17x^2-4x+1}{5x^2+9x-2}=12x-25+\dfrac{49}{x+2}$.

16. Shew that $1+\dfrac{a^2+b^2-c^2}{2ab}=\dfrac{(a+b+c)(a+b-c)}{2ab}$.

17. Divide $x+\dfrac{16x-27}{x^2-16}$ by $x-1+\dfrac{13}{x+4}$.

18. Multiply $a^2-2ax+4x^2-\dfrac{16x^3}{a+2x}$ by $3-\dfrac{6x(a+4x)}{a^2+2ax+4x^2}$.

19. Divide $b^2+3b-2-\dfrac{12}{b-3}$ by $3b+6-\dfrac{2b^2}{b-3}$.

20. Divide $a^2+9b^2+\dfrac{65b^4}{a^2-9b^2}$ by $a+3b+\dfrac{13b^2}{a-3b}$.

21. Multiply $4x^2+14x+\dfrac{98x-27}{2x-7}$ by $\dfrac{1}{6}-\dfrac{3x+29}{12x^2+18x+27}$.

185. We add a Miscellaneous Exercise in which most of the processes connected with fractions will be illustrated.

## EXAMPLES XXII. c.

Simplify the following fractions:

1. $\dfrac{4a(a^2-x^2)}{3b(c^2-x^2)} \div \left[\dfrac{a^2-ax}{bc+bx} \times \dfrac{a^2+2ax+x^2}{c^2-2cx+x^2}\right]$.

2. $\dfrac{x(x+a)(x+2a)}{3a} - \dfrac{x(x+a)(2x+a)}{6a}$.

3. $\dfrac{1}{b}\left(\dfrac{1}{a-b} - \dfrac{1}{a+2b}\right) - \dfrac{2}{a^2+ab-2b^2}$. 4. $\left(\dfrac{x+y}{x-y}\right)^2 - \left(\dfrac{x-y}{x+y}\right)^2$.

5. $\dfrac{2}{x-1} + \dfrac{2}{x+1} - \dfrac{4x}{x^2-x+1}$.

6. $\left(\dfrac{x^2}{1-x^4} + \dfrac{2x^4}{1-x^8}\right) \div \left(\dfrac{x^2+1}{x}\right)^2$.

7. $\dfrac{1}{x} - \dfrac{1}{(x+1)^2} - \dfrac{2}{x+1} + \dfrac{x}{1+x+x^2}$.

8. $\dfrac{1+x^3}{1+2x+2x^2+x^3}$. 9. $\dfrac{2x^3-9x^2+27}{3x^3-81x+162}$.

10. $\dfrac{a}{b} - \dfrac{(a^2-b^2)x}{b^2} + \dfrac{a(a^2-b^2)x^2}{b^2(b+ax)}$.

11. $\left\{\dfrac{x^4-a^4}{x^2-2ax+a^2} \div \dfrac{x^2+ax}{x-a}\right\} \times \dfrac{x^5-a^2x^3}{x^3+a^3} \div \left(\dfrac{x}{a} - \dfrac{a}{x}\right)$.

12. $\dfrac{a^2-x^2}{a^2+ax+x^2} \div \dfrac{\left(1-\dfrac{x}{a}\right)^3\left(1+\dfrac{x}{a}\right)^3}{a^3-x^3}$.

13. $\dfrac{x^4-2x^2+1}{3x^5-10x^3+15x-8}$. 14. $\dfrac{a^3+a(1+a)y+y^2}{a^4-y^2}$.

15. $\dfrac{1}{a} + \dfrac{2}{a+1} + \dfrac{3}{a+2} - \dfrac{\dfrac{4}{a}}{1+\dfrac{1}{a}}$.

**16.** $\dfrac{x+3}{2x^2+9x+9}+\dfrac{1}{2}\cdot\dfrac{1}{2x-3}-\dfrac{1}{x-\dfrac{9}{4x}}$.

**17.** $\dfrac{2}{x^3+x^2+x+1}-\dfrac{2}{x^3-x^2+x-1}$.

**18.** $\dfrac{1-a^2}{(1+ax)^2-(a+x)^2}\div\dfrac{1}{2}\left(\dfrac{1}{1-x}+\dfrac{1}{1+x}\right)$.

**19.** $\dfrac{2x^3-x^2-2x+1}{x^3-3x+2}$.

**20.** $\dfrac{x^2-6x+8}{4x^3-21x^2+15x+20}$.

**21.** $\dfrac{2a}{(x-2a)^2}-\dfrac{x-a}{x^2-5ax+6a^2}+\dfrac{2}{x-3a}$.

**22.** $\dfrac{1}{2}\left(\dfrac{a^2+x^2}{a^2-x^2}\right)-\dfrac{1}{2}\cdot\dfrac{a+x}{a-x}-\left(\dfrac{a}{a+x}\right)^2$.

**23.** $\dfrac{x}{2}\left(\dfrac{1}{x-y}-\dfrac{1}{x+y}\right)\times\dfrac{x^2-y^2}{x^2y+xy^2}\div\dfrac{1}{x+y}$.

**24.** $\dfrac{1}{x+y}\div\left[\dfrac{y}{2}\left(\dfrac{1}{x+y}+\dfrac{1}{x-y}\right)\times\dfrac{x^2-y^2}{x^2y+xy^2}\right]$.

**25.** $\left(3x-5-\dfrac{2}{x}\right)\left(3x+5-\dfrac{2}{x}\right)\div\left(x-\dfrac{4}{x}\right)$.

**26.** $\left\{\dfrac{2}{x}-\dfrac{1}{a+x}+\dfrac{1}{a-x}\right\}\div\left(\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right)$.

**27.** $\dfrac{1}{2x-1}-\dfrac{2x-\dfrac{1}{2x}}{4x^2-1}$.

**28.** $\left(b+\dfrac{ab}{b-a}\right)\left(b-\dfrac{ab}{a+b}\right)\left(\dfrac{b^2-a^2}{b^2+a^2}\right)$.

**29.** $\left\{\dfrac{b+\dfrac{a-b}{1+ab}}{1-\dfrac{(a-b)b}{1+ab}}-\dfrac{a-\dfrac{a-b}{1-ab}}{1-\dfrac{a(a-b)}{1-ab}}\right\}\div\left(\dfrac{a}{b}-\dfrac{b}{a}\right)$.

**30.** $\dfrac{\dfrac{x^2+y^2}{y}-x}{\dfrac{1}{y}-\dfrac{1}{x}}\times\dfrac{x^2-y^2}{x^3+y^3}$.

**31.** $\dfrac{2\left(x^2-\dfrac{1}{4}\right)}{2x+1}+\dfrac{1}{2}$.

**32.** $$\frac{a+\dfrac{b-a}{1+ab}}{1-\dfrac{a(b-a)}{1+ab}}\times\frac{\dfrac{x+y}{1-xy}-y}{1+\dfrac{y(x+y)}{1-xy}}.$$

**33.** $$\frac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a-b}{a+b}-\dfrac{a+b}{a-b}}\times\frac{ab^3-a^3b}{a^2+b^2}.$$

**34.** $$\frac{(1-x^2)(1-x^3)}{x(1+x)(1-x)^2}-\frac{x^3+\dfrac{1}{x^3}}{x^2+\dfrac{1}{x^2}-1}.$$

**35.** $$\left\{x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right)\right\}\div\left(x-\frac{1}{x}\right).$$

**36.** $$\frac{1+\dfrac{1}{m}}{\dfrac{1}{m}}\times\frac{\dfrac{1}{m}}{m^2+\dfrac{1}{m}}\div\frac{\dfrac{1}{m}}{m-1+\dfrac{1}{m}}.$$

**37.** $$\frac{\dfrac{x}{y}+\dfrac{y}{x}-1}{\dfrac{x^2}{y^2}+\dfrac{x}{y}+1}\times\frac{1+\dfrac{y}{x}}{x-y}\div\frac{1+\dfrac{y^3}{x^3}}{\dfrac{x^2}{y}-\dfrac{y^2}{x}}.$$

**38.** $$\frac{\left(\dfrac{3x+x^3}{1+3x^2}\right)^2-1}{\dfrac{3x^2-1}{x^3-3x}+1}\div\frac{\dfrac{9}{x^2}-\dfrac{33-x^2}{3x^2+1}}{\dfrac{3}{x^2}-\dfrac{2(x^2+3)}{(x^3-x)^2}}.$$

**39.** $$\frac{1}{a^2-2}-\frac{2}{a^2-1}+\frac{2}{a^2+1}-\frac{1}{a^2+2}.$$

**40.** $$\frac{1}{6m-2n}+\frac{1}{3m+2n}-\frac{3}{6m+2n}.$$

**41.** $$\frac{3}{4(1-x)^2}+\frac{3}{8(1-x)}+\frac{1}{8(1+x)}+\frac{x-1}{4(1+x^2)}.$$

**42.** $$\frac{4}{9(x-2)}+\frac{5}{9(x+1)}-\frac{1}{3(x+1)^2}-\frac{1}{x+2+\dfrac{1}{x}}.$$

43. $\left(\frac{x^2}{y}+\frac{y^2}{x}\right)\left(\frac{1}{y^2-x^2}\right)-\frac{y}{x^2+xy}+\frac{x}{xy-y^2}.$

44. $\frac{x^2-(y-z)^2}{(x+z)^2-y^2}+\frac{y^2-(z-x)^2}{(x+y)^2-z^2}+\frac{z^2-(x-y)^2}{(y+z)^2-x^2}.$

45. $\frac{x^2-(y-2z)^2}{(2z+x)^2-y^2}+\frac{y^2-(2z-x)^2}{(x+y)^2-4z^2}+\frac{4z^2-(x-y)^2}{(y+2z)^2-x^2}.$

46. $\frac{(x-y)(y-z)+(y-z)(z-x)+(z-x)(x-y)}{x(z-x)+y(x-y)+z(y-z)}.$

47. $\frac{a-b-c}{(a-b)(a-c)}+\frac{b-c-a}{(b-c)(b-a)}+\frac{c-a-b}{(c-a)(c-b)}.$

48. $\frac{c+a}{(a-b)(a-c)}+\frac{a+b}{(b-c)(b-a)}+\frac{b+c}{(c-a)(c-b)}.$

49. $\frac{x^2-(2y-3z)^2}{(3z+x)^2-4y^2}+\frac{4y^2-(3z-x)^2}{(x+2y)^2-9z^2}+\frac{9z^2-(x-2y)^2}{(2y+3z)^2-x^2}.$

50. $\frac{9y^2-(4z-2x)^2}{(2x+3y)^2-16z^2}+\frac{16z^2-(2x-3y)^2}{(3y+4z)^2-4x^2}+\frac{4x^2-(3y-4z)^2}{(4z+2x)^2-9y^2}.$

51. $\dfrac{\frac{1}{x}-\frac{x+a}{x^2+a^2}}{\frac{1}{a}-\frac{a+x}{a^2+x^2}}+\dfrac{\frac{1}{x}-\frac{x-a}{x^2+a^2}}{\frac{1}{a}-\frac{a-x}{a^2+x^2}}.$

52. $\frac{(x+a)(x+b)-(y+a)(y+b)}{x-y}-\frac{(x-a)(y-b)-(x-b)(y-a)}{a-b}.$

53. $\left(\frac{a+x}{a^2-ax+x^2}-\frac{a-x}{a^2+ax+x^2}\right)\div\left(\frac{a^2+x^2}{a^3-x^3}-\frac{a^2-x^2}{a^3+x^3}\right).$

54. $\dfrac{\frac{x-1}{3}+\frac{x-1}{x-2}}{\frac{x+2}{4}+\frac{x+2}{x-3}}\div\dfrac{\frac{x+3}{7}-\frac{x+3}{x+4}}{\frac{x-2}{3}+\frac{x-2}{x-1}}.$

# CHAPTER XXIII.

## HARDER EQUATIONS.

186. IN this chapter we propose to give a miscellaneous collection of equations. Some of these will serve as a useful exercise for revision of the methods already explained in previous chapters; but we also add others presenting more difficulty, the solution of which will often be facilitated by some special artifice.

The following examples worked in full will sufficiently illustrate the most useful methods.

*Example* 1. Solve $\frac{6x-3}{2x+7}=\frac{3x-2}{x+5}$.

Multiplying up, we have

$$(6x-3)(x+5)=(3x-2)(2x+7),$$
$$6x^2+27x-15=6x^2+17x-14;$$
$$\therefore\ 10x=1;$$
$$\therefore\quad x=\frac{1}{10}.$$

NOTE. By a simple reduction many equations can be brought to the form in which the above equation is given. When this is the case, the necessary simplification is readily completed by multiplying up, or "multiplying across", as it is sometimes called.

*Example* 2. Solve $\frac{8x+23}{20}-\frac{5x+2}{3x+4}=\frac{2x+3}{5}-1$.

Multiply by 20, and we have

$$8x+23-\frac{20(5x+2)}{3x+4}=8x+12-20.$$

By transposition, $$31=\frac{20(5x+2)}{3x+4}.$$

Multiplying across, $93x+124=20(5x+2)$,

$$84=7x;$$
$$\therefore\ x=12.$$

When two or more fractions have the same denominator they should be taken together and simplified.

*Example* 3. Solve $\frac{13-2x}{x+3}+\frac{23x+8\frac{1}{3}}{4x+5}=\frac{16-\frac{1}{4}x}{x+3}+4.$

By transposition, we have

$$\frac{23x+8\frac{1}{3}}{4x+5}-4=\frac{16-\frac{1}{4}x-13+2x}{x+3};$$

$$\therefore\ \frac{7x-\frac{35}{3}}{4x+5}=\frac{3+\frac{7x}{4}}{x+3}.$$

Multiplying across, we have

$$7x^2-\frac{35x}{3}+21x-35=12x+7x^2+15+\frac{35x}{4},$$

$$-\frac{137x}{12}=50;$$

$$\therefore\ x=-\frac{600}{137}.$$

*Example* 4. Solve $\frac{x-8}{x-10}+\frac{x-4}{x-6}=\frac{x-5}{x-7}+\frac{x-7}{x-9}.$

This equation might be solved by clearing of fractions, but the work would be very laborious. The solution will be much simplified by proceeding as follows:

Transposing, $\frac{x-8}{x-10}-\frac{x-5}{x-7}=\frac{x-7}{x-9}-\frac{x-4}{x-6}.$

Simplifying each side *separately*, we have

$$\frac{(x-8)(x-7)-(x-5)(x-10)}{(x-10)(x-7)}=\frac{(x-7)(x-6)-(x-4)(x-9)}{(x-9)(x-6)};$$

$$\therefore\ \frac{x^2-15x+56-(x^2-15x+50)}{(x-10)(x-7)}=\frac{x^2-13x+42-(x^2-13x+36)}{(x-9)(x-6)};$$

$$\therefore\ \frac{6}{(x-10)(x-7)}=\frac{6}{(x-9)(x-6)}.$$

Hence, since the numerators are equal, the denominators must be equal;

that is, $$(x-10)(x-7)=(x-9)(x-6),$$

$$x^2-17x+70=x^2-15x+54;$$

$$\therefore\ 16=2x;$$

$$\therefore\ x=8.$$

The above equation may also be solved very neatly by the following artifice.

The equation may be written in the form

$$\frac{(x-10)+2}{x-10}+\frac{(x-6)+2}{x-6}=\frac{(x-7)+2}{x-7}+\frac{(x-9)+2}{x-9};$$

whence we have

$$1+\frac{2}{x-10}+1+\frac{2}{x-6}=1+\frac{2}{x-7}+1+\frac{2}{x-9};$$

which gives $$\frac{1}{x-10}+\frac{1}{x-6}=\frac{1}{x-7}+\frac{1}{x-9}.$$

Transposing, $$\frac{1}{x-10}-\frac{1}{x-7}=\frac{1}{x-9}-\frac{1}{x-6};$$

$$\therefore \frac{3}{(x-10)(x-7)}=\frac{3}{(x-9)(x-6)}$$

and the solution may be completed as before.

*Example* 5. Solve $\frac{5x-64}{x-13}-\frac{2x-11}{x-6}=\frac{4x-55}{x-14}-\frac{x-6}{x-7}$.

We have $5+\frac{1}{x-13}-\left(2+\frac{1}{x-6}\right)=4+\frac{1}{x-14}-\left(1+\frac{1}{x-7}\right)$;

$$\therefore \frac{1}{x-13}-\frac{1}{x-6}=\frac{1}{x-14}-\frac{1}{x-7}.$$

Simplifying each side *separately*, we have

$$\frac{7}{(x-13)(x-6)}=\frac{7}{(x-14)(x-7)},$$

$$(x-13)(x-6)=(x-14)(x-7),$$

$$x^2-19x+78=x^2-21x+98,$$

$$2x=20;$$

$$\therefore x=10.$$

## EXAMPLES XXIII. a.

1. $\frac{x+4}{3x-8}=\frac{x+5}{3x-7}$.

2. $\frac{3x+1}{3(x-2)}=\frac{x-2}{x-1}$.

3. $\frac{7-5x}{1+x}=\frac{11-15x}{1+3x}$.

4. $\frac{3(7+6x)}{2+9x}=\frac{35+4x}{9+2x}$.

5. $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}$.

6. $\frac{6x+8}{2x+1}-\frac{2x+38}{x+12}=1$.

**7.** $\frac{3x-1}{2x-1}-\frac{4x-2}{3x-1}=\frac{1}{6}.$

**8.** $\frac{x+25}{x-5}=\frac{2x+75}{2x-15}.$

**9.** $\frac{x}{x+2}+\frac{4}{x+6}=1.$

**10.** $\frac{6x+7}{9x+6}=\frac{1}{12}+\frac{5x-5}{12x+8}.$

**11.** $\frac{2x-5}{5}+\frac{x-3}{2x-15}=\frac{4x-3}{10}-1\tfrac{1}{10}.$

**12.** $\frac{4(x+3)}{9}=\frac{8x+37}{18}-\frac{7x-29}{5x-12}.$

**13.** $\frac{(2x-1)(3x+8)}{6x(x+4)}-1=0.$

**14.** $\frac{2x+5}{5x+3}-\frac{2x+1}{5x+2}=0.$

**15.** $\frac{4}{x+3}-\frac{2}{x+1}=\frac{5}{2x+6}-\frac{2\frac{1}{2}}{2x+2}.$

**16.** $\frac{7}{x-4}-\frac{60}{5x-30}=\frac{10\frac{1}{2}}{3x-12}-\frac{8}{x-6}.$

**17.** $\frac{3}{4-2x}+\frac{30}{8(1-x)}=\frac{3}{2-x}+\frac{5}{2-2x}.$

**18.** $\frac{25-\frac{x}{3}}{x+1}+\frac{16x+4\frac{1}{5}}{3x+2}=5+\frac{23}{x+1}.$

**19.** $\frac{30+6x}{x+1}+\frac{60+8x}{x+3}=14+\frac{48}{x+1}.$

**20.** $\frac{x}{x-2}-\frac{x+1}{x-1}=\frac{x-8}{x-6}-\frac{x-9}{x-7}.$

**21.** $\frac{x+5}{x+4}-\frac{x-6}{x-7}=\frac{x-4}{x-5}-\frac{x-15}{x-16}.$

**22.** $\frac{x-7}{x-9}-\frac{x-9}{x-11}=\frac{x-13}{x-15}-\frac{x-15}{x-17}.$

**23.** $\frac{x+3}{x+6}-\frac{x+6}{x+9}=\frac{x+2}{x+5}-\frac{x+5}{x+8}.$

**24.** $\frac{x+2}{x}+\frac{x-7}{x-5}-\frac{x+3}{x+1}=\frac{x-6}{x-4}.$

**25.** $\frac{4x-17}{x-4}+\frac{10x-13}{2x-3}=\frac{8x-30}{2x-7}+\frac{5x-4}{x-1}.$

**26.** $\frac{5x-8}{x-2}+\frac{6x-44}{x-7}-\frac{10x-8}{x-1}=\frac{x-8}{x-6}.$

**27.** $\dfrac{2x-3}{\cdot 3x-\cdot 4}=\dfrac{\cdot 4x-\cdot 6}{\cdot 06x-\cdot 07}$. **28.** $\dfrac{x-2}{\cdot 0\dot{5}}-\dfrac{x-4}{\cdot 0625}=56.$

**29.** $\cdot 08\dot{3}\,(x-\cdot 625)=\cdot\dot{0}\dot{9}\,(x-\cdot 59375).$

**30.** $(2x+1\cdot 5)(3x-2\cdot 25)=(2x-1\cdot 125)(3x+1\cdot 25).$

**31.** $\dfrac{\cdot 3x-1}{\cdot 5x-\cdot 4}=\dfrac{\cdot 5+1\cdot 2x}{2x-\cdot 1}$. **32.** $\dfrac{1-1\cdot 4x}{\cdot 2+x}=\dfrac{\cdot 7\,(x-1)}{\cdot 1-\cdot 5x}$.

**33.** $\dfrac{(\cdot 3x-2)(\cdot 3x-1)}{\cdot 2x-1}-\dfrac{1}{6}(\cdot 3x-2)=\cdot 4x-2.$

## Literal Equations.

187. In the equations we have discussed hitherto the coefficients have been numerical quantities, but equations often involve *literal* coefficients. [Art. 6.] These are supposed to be known, and will appear in the solution.

*Example* 1. Solve $(x+a)(x+b)-c(a+c)=(x-c)(x+c)+ab.$

Multiplying out, we have

$$x^2+ax+bx+ab-ac-c^2=x^2-c^2+ab;$$

Whence

$$ax+bx=ac,$$

$$(a+b)x=ac;$$

$$\therefore\ x=\frac{ac}{a+b}.$$

*Example* 2. Solve $\dfrac{a}{x-a}-\dfrac{b}{x-b}=\dfrac{a-b}{x-c}$.

Simplifying the left side, we have

$$\frac{a(x-b)-b(x-a)}{(x-a)(x-b)}=\frac{a-b}{x-c},$$

$$\frac{(a-b)x}{(x-a)(x-b)}=\frac{a-b}{x-c};$$

$$\therefore\ \frac{x}{(x-a)(x-b)}=\frac{1}{x-c}.$$

Multiplying across, $x^2-cx=x^2-ax-bx+ab,$

$$ax+bx-cx=ab,$$

$$(a+b-c)x=ab;$$

$$\therefore\ x=\frac{ab}{a+b-c}.$$

## EXAMPLES XXIII. b.

1. $ax-2b=5bx-3a.$

2. $a^2(x-a)+b^2(x-b)=abx.$

3. $x^2+a^2=(b-x)^2.$

4. $(x-a)(x+b)=(x-a+b)^2.$

5. $a(x-2)+2x=6+a.$

6. $m^2(m-x)-mnx=n^2(n+x).$

7. $(a+x)(b+x)=x(x-c).$

8. $(a-b)(x-a)=(a-c)(x-b).$

9. $\dfrac{2x+3a}{x+a}=\dfrac{2(3x+2a)}{3x+a}.$

10. $\dfrac{2(x-b)}{3x-c}=\dfrac{2x+b}{3(x-c)}.$

11. $\dfrac{1}{a}-\dfrac{1}{x}=\dfrac{1}{x}-\dfrac{1}{b}.$

12. $\dfrac{2}{3}\left(\dfrac{x}{a}+1\right)=\dfrac{3}{4}\left(\dfrac{x}{a}-1\right).$

13. $\dfrac{a}{x}=c(a-b)+\dfrac{b}{x}.$

14. $\dfrac{9a}{b}-\dfrac{3x}{b}=\dfrac{4b}{a}-\dfrac{2x}{a}.$

15. $\dfrac{x-a}{b-x}=\dfrac{x-b}{a-x}.$

16. $\dfrac{x-a}{2}=\dfrac{(x-b)^2}{2x-a}.$

17. $\dfrac{1}{4}x(x-a)-\left(\dfrac{x+a}{2}\right)^2=\dfrac{2a}{3}\left(x-\dfrac{a}{2}\right).$

18. $(a+b)x^2-a(bx+a^2)=bx(x-a)+ax(x-b).$

19. $b(a+x)-(a+x)(b-x)=x^2+\dfrac{bc^2}{a}.$

20. $b(a-x)-\dfrac{a}{b}(b+x)^2+ab\left(\dfrac{x}{b}+1\right)^2=0.$

21. $x^2+a(2a-x)-\dfrac{3b^2}{4}=\left(x-\dfrac{b}{2}\right)^2+a^2.$

22. $(2x-a)\left(x+\dfrac{2a}{3}\right)=4x\left(\dfrac{a}{3}-x\right)-\dfrac{1}{2}(a-4x)(2a+3x).$

*Example* 3. Solve $ax+by=c$ ................................(1),

$a'x+b'y=c'$ ................................ (2).

The notation here first used is one that the student will frequently meet with in the course of his reading. In the first equation we choose certain letters as the coefficients of $x$ and $y$, and we choose *corresponding letters with accents* to denote corresponding quantities in the second equation. There is no necessary connection between the values of $a$ and $a'$, and they are as different as $a$ and $b$; but it is often convenient to use the same letter thus slightly varied to mark some common meaning of such letters, and thereby assist the memory. Thus $a$, $a'$ have a common property as being coefficients of $x$; $b$, $b'$ as being coefficients of $y$.

Sometimes instead of accents letters are used with a *suffix*, such as $a_1$, $a_2$, $a_3$; $b_1$, $b_2$, $b_3$, &c.

To return to the equations $ax+by=c$ ............................ (1),

$$a'x+b'y=c' \text{ ............................ } (2).$$

Multiply (1) by $b'$ and (2) by $b$. Thus

$$ab'x+bb'y=b'c,$$

$$a'bx+bb'y=bc',$$

by subtraction, $$(ab'-a'b)\,x=b'c-bc';$$

$$\therefore\ x=\frac{b'c-bc'}{ab'-a'b} \text{ .................... } (3).$$

As previously explained in Art. 104, we might obtain $y$ by substituting this value of $x$ in *either* of the equations (1) or (2); but $y$ is more conveniently found by eliminating $x$, as follows:

Multiplying (1) by $a'$ and (2) by $a$, we have

$$aa'x+a'by=a'c,$$

$$aa'x+ab'y=ac',$$

by subtraction, $$(a'b-ab')\,y=a'c-ac';$$

$$\therefore\ y=\frac{a'c-ac'}{a'b-ab'},$$

or, changing signs in the terms of the denominator so as to have the same denominator as in (3),

$$y=\frac{ac'-a'c}{ab'-a'b}, \quad \text{and } x=\frac{b'c-bc'}{ab'-a'b}.$$

*Example* 4. Solve $$\frac{x-a}{c-a}+\frac{y-b}{c-b}=1 \text{ .......................... } (1),$$

$$\frac{x+a}{c}+\frac{y-a}{a-b}=\frac{a}{c} \text{ .......................... } (2).$$

From (1) by clearing of fractions, we have

$$x(c-b)-a(c-b)+y(c-a)-b(c-a)=(c-a)(c-b),$$

$$x(c-b)+y(c-a)=ac-ab+bc-ab+c^2-ac-bc+ab,$$

$$x(c-b)+y(c-a)=c^2-ab \text{ .................... } (3).$$

Again, from (2), we have

$$x(a-b)+a(a-b)+cy-ca=a(a-b)$$

$$x(a-b)+cy=ac \text{ .................... } (4).$$

Multiply (3) by $c$ and (4) by $c-a$ and subtract,

$$x\{c(c-b)-(c-a)(a-b)\}=c^3-abc-ac(c-a),$$

$$x(c^2-ac+a^2-ab)=c(c^2-ab-ac+a^2);$$

$$\therefore\ x=c;$$

and therefore from (4) $$y=b.$$

## EXAMPLES XXIII. c.

**1.** $ax+by=l$
$bx+ay=m.$

**2.** $lx+my=n$
$px+qy=r.$

**3.** $ax=by$
$bx+ay=c.$

**4.** $ax+by=a^2$
$bx+ay=b^2.$

**5.** $x+ay=a'$
$ax+a'y=1.$

**6.** $px-qy=r$
$rx-py=q.$

**7.** $\frac{x}{a}+\frac{y}{b}=\frac{1}{ab}$
$\frac{x}{a'}-\frac{y}{b'}=\frac{1}{a'b'}.$

**8.** $\frac{x}{a}-\frac{y}{b}=0$
$bx+ay=4ab.$

**9.** $\frac{3x}{a}+\frac{2y}{b}=3$
$\frac{9x}{a}-\frac{6y}{b}=3$

**10.** $qx-rb=p(a-y)$
$\frac{qx}{a}+r=p\left(1+\frac{y}{b}\right).$

**11.** $\frac{x}{m}+\frac{y}{m'}=1$
$\frac{x}{m'}-\frac{y}{m}=1.$

**12.** $px+qy=0$
$lx+my=n.$

**13.** $(a-b)x=(a+b)y$
$x+y=c.$

**14.** $(a-b)x+(a+b)y=2a^2-2b^2$
$(a+b)x-(a-b)y=4ab.$

**15.** $\frac{x}{a}+\frac{y}{b}=1$
$\frac{x}{3a}+\frac{y}{6b}=\frac{2}{3}.$

**16.** $\frac{x}{a}+\frac{y}{b}=2$
$\frac{x}{a'}=\frac{y}{b'}.$

**17.** $\frac{x}{a}-\frac{y}{b}=1$
$\frac{x}{b}+\frac{y}{a}=\frac{a}{b}.$

**18.** $\frac{m}{l}x+\frac{l}{m}y=\left(\frac{1}{l}+\frac{1}{m}\right)(m^2+l^2)$
$(x+y)(m^2+l^2)=2(m^3+l^3)+ml(x+y).$

**19.** $bx+cy=a+b$
$ax\left(\frac{1}{a-b}-\frac{1}{a+b}\right)+cy\left(\frac{1}{b-a}-\frac{1}{b+a}\right)=\frac{2a}{a+b}.$

**20.** $(a-b)x+(a+b)y=2(a^2-b^2)$
$ax-by=a^2+b^2.$

# CHAPTER XXIV.

## HARDER PROBLEMS.

188. In previous chapters we have given collections of problems which lead to simple equations. We add here a few examples of somewhat greater difficulty.

*Example* 1. A grocer buys 15 lbs. of figs and 28 lbs. of currants for £1. 1*s*. 8*d*.; by selling the figs at a loss of 10 per cent., and the currants at a gain of 30 per cent., he clears 2*s*. 6*d*. on his outlay; how much per pound did he pay for each?

Let $x$, $y$ denote the number of pence in the price of a pound of figs and currants respectively; then the outlay is

$$15x + 28y \text{ pence.}$$

Therefore $$15x + 28y = 260 \quad \dots\dots\dots\dots(1).$$

The loss upon the figs is $\frac{1}{10} \times 15x$ pence, and the gain upon the currants is $\frac{3}{10} \times 28y$ pence; therefore the total gain is

$$\frac{42y}{5} - \frac{3x}{2} \text{ pence;}$$

$$\therefore \frac{42y}{5} - \frac{3x}{2} = 30 \quad \dots\dots\dots\dots(2).$$

From (1) and (2) we find that $x = 8$, and $y = 5$; that is the figs cost 8*d*. a pound, and the currants cost 5*d*. a pound.

*Example* 2. At what time between 4 and 5 o'clock will the minute-hand of a watch be 13 minutes in advance of the hour-hand?

Let $x$ denote the required number of minutes after 4 o'clock; then, as the minute-hand travels twelve times as fast as the hour-hand, the hour-hand will move over $\frac{x}{12}$ minute divisions in $x$ minutes. At 4 o'clock the minute-hand is 20 divisions behind the hour-hand, and finally the minute-hand is 13 divisions in advance; therefore the minute-hand moves over 20 + 13, or 33 divisions more than the hour-hand.

Hence $$x=\frac{x}{12}+33,$$
$$\frac{11}{12}x=33;$$
$$\therefore\ x=36.$$

Thus the time is 36 minutes past 4.

If the question be asked as follows: "At what *times* between 4 and 5 o'clock will there be 13 minutes between the two hands?" we must also take into consideration the case when the minute-hand is 13 divisions *behind* the hour-hand. In this case the minute-hand gains 20 − 13, or 7 divisions.

Hence $$x=\frac{x}{12}+7,$$

which gives $$x=7\frac{7}{11}.$$

Therefore the *times* are $7\frac{7'}{11}$ past 4, and 36′ past 4.

*Example* 3. Two persons $A$ and $B$ start simultaneously from two places, $c$ miles apart, and walk in the same direction. $A$ travels at the rate of $p$ miles an hour, and $B$ at the rate of $q$ miles; how far will $A$ have walked before he overtakes $B$?

Suppose $A$ has walked $x$ miles, then $B$ has walked $x-c$ miles.

$A$ walking at the rate of $p$ miles an hour will travel $x$ miles in $\frac{x}{p}$ hours; and $B$ will travel $x-c$ miles in $\frac{x-c}{q}$ hours: these two times being equal, we have

$$\frac{x}{p}=\frac{x-c}{q},$$
$$qx=px-pc;$$

whence $$x=\frac{pc}{p-q}.$$

Therefore $A$ has travelled $\frac{pc}{p-q}$ miles.

*Example* 4. A train travelled a certain distance at a uniform rate. Had the speed been 6 miles an hour more, the journey would have occupied 4 hours less; and had the speed been 6 miles an hour less, the journey would have occupied 6 hours more. Find the distance.

Let the speed of the train be $x$ miles per hour, and let the time occupied be $y$ hours; then the distance traversed will be represented by $xy$ miles.

On the first supposition the speed per hour is $x+6$ miles, and the time taken is $y-4$ hours. In this case the distance traversed will be represented by $(x+6)(y-4)$ miles.

On the second supposition the distance traversed will be represented by $(x-6)(y+6)$ miles.

All these expressions for the distance must be equal;

$$\therefore\ xy=(x+6)(y-4)=(x-6)(y+6).$$

From these equations we have

$$xy=xy+6y-4x-24,$$

or $$6y-4x=24 \quad\ldots\ldots(1);$$

and $$xy=xy-6y+6x-36,$$

or $$6x-6y=36 \quad\ldots\ldots(2).$$

From (1) and (2) we obtain $x=30$, $y=24$.

Hence the distance is 720 miles.

*Example* 5. A person invests £3770, partly in 3 per Cent. Consols at £102, and partly in Railway Stock at £84 which pays a dividend of $4\frac{1}{2}$ per cent.: if his income from these investments is £136. 5*s.* per annum, what sum does he invest in each?

Let $x$ denote the number of pounds invested in Consols, $y$ the number of pounds invested in Railway Stock; then

$$x+y=3770 \quad\ldots\ldots(1).$$

The income from Consols is £$\frac{3x}{102}$, or £$\frac{x}{34}$; and that from Railway Stock is £$\frac{4\frac{1}{2}y}{84}$, or £$\frac{3y}{56}$.

Therefore $$\frac{x}{34}+\frac{3y}{56}=136\tfrac{1}{4} \quad\ldots\ldots(2).$$

From (2) $$x+\frac{51}{28}y=4632\tfrac{1}{2};$$

therefore by subtracting (1)

$$\frac{23}{28}y=862\tfrac{1}{2};$$

whence $$y=28\times 37\tfrac{1}{2}=1050;$$

and from (1) $$x=2720.$$

Therefore he invests £2720 in Consols, and £1050 in Railway Stock.

## EXAMPLES XXIV.

**1.** A sum of £10 is divided among a number of persons; if the number had been increased by one-fourth each would have received a shilling less: find the number of persons.

2. I bought a certain number of eggs at four a penny; I kept one-fifth of them, and sold the rest at three a penny, and gained a penny: how many did I buy?

3. I bought a certain number of articles at five for sixpence; if they had been eleven for one shilling, I should have spent sixpence less: how many did I buy?

4. A man at whist wins twice as much as he had to begin with, and then loses 16*s.*; he then loses four-fifths of what remained, and afterwards wins as much as he had at first: how much had he originally, if he leaves off with £4?

5. I spend £14. 5*s.* in buying 20 yards of calico and 30 yards of silk; the silk costs as many shillings per yard as the calico costs pence per yard: find the price of each.

6. A number of two digits exceeds five times the sum of its digits by 9, and its ten-digit exceeds its unit-digit by 1: find the number.

7. The sum of the digits of a number less than 100 is 6; if the digits be reversed the resulting number will be less by 18 than the original number: find it.

8. A man being asked his age replied, "If you take 2 years from my present age the result will be double my wife's age, and 3 years ago her age was one-third of what mine will be in 12 years." What were their ages?

9. At what time between one and two o'clock are the hands of a watch first at right angles?

10. At what time between 3 and 4 o'clock is the minute-hand one minute ahead of the hour-hand?

11. When are the hands of a clock together between the hours of 6 and 7?

12. It is between 2 and 3 o'clock, and in 10 minutes the minute-hand will be as much before the hour-hand as it is now behind it: what is the time?

13. At an election the majority was 162, which was three-elevenths of the whole numbers of voters: what was the number of the votes on each side?

14. A certain number of persons paid a bill; if there had been ten more each would have paid 2*s.* less; if there had been 5 less each would have paid 2*s.* 6*d.* more: find the number of persons, and what each had to pay.

15. A man spends £5 in buying two kinds of silk at 4*s.* 6*d.* and 4*s.* a yard; by selling it all at 4*s.* 3*d.* per yard he gains 2 per cent.: how much of each did he buy?

**16.** Ten years ago the sum of the ages of two sons was one-third of their father's age: one is two years older than the other, and the present sum of their ages is fourteen years less than their father's age: how old are they?

**17.** *A* and *B* start from the same place walking at different rates; when *A* has walked 15 miles *B* doubles his pace, and 6 hours later passes *A*: if *A* walks at the rate of 5 miles an hour, what is *B*'s rate at first?

**18.** A basket of oranges is emptied by one person taking half of them and one more, a second person taking half of the remainder and one more, and a third person taking half of the remainder and six more. How many did the basket contain at first?

**19.** A person swimming in a stream which runs $1\frac{1}{2}$ miles per hour, finds that it takes him four times as long to swim a mile up the stream as it does to swim the same distance down: at what rate does he swim?

**20.** At what *times* between 7 and 8 o'clock will the hands of a watch be at right angles to each other? When will they be in the same straight line?

**21.** The denominator of a fraction exceeds the numerator by 4; and if 5 is taken from each, the sum of the reciprocal of the new fraction and four times the original fraction is 5: find the original fraction.

**22.** Two persons start at noon from towns 60 miles apart. One walks at the rate of four miles an hour, but stops $2\frac{1}{2}$ hours on the way; the other walks at the rate of 3 miles an hour without stopping: when and where will they meet?

**23.** *A*, *B*, and *C* travel from the same place at the rates of 4, 5, and 6 miles an hour respectively; and *B* starts 2 hours after *A*. How long after *B* must *C* start in order that they may overtake *A* at the same instant?

**24.** A dealer bought a horse, expecting to sell it again at a price that would have given him 10 per cent. profit on his purchase; but he had to sell it for £50 less than he expected, and he then found that he had lost 15 per cent. on what it cost him: what did he pay for the horse?

**25.** A man walking from a town, *A*, to another, *B*, at the rate of 4 miles an hour, starts one hour before a coach travelling 12 miles an hour, and is picked up by the coach. On arriving at *B*, he finds that his coach journey has lasted 2 hours: find the distance between *A* and *B*.

**26.** What is the property of a person whose income is £1140, when one-twelfth of it is invested at 2 per cent., one-half at 3 per cent., one-third at $4\frac{1}{2}$ per cent., and the remainder pays him no dividend?

**27.** A person spends one-third of his income, saves one-fourth, and pays away 5 per cent. on the whole as interest at $7\frac{1}{2}$ per cent. on debts previously incurred, and then has £110 remaining: what was the amount of his debts?

**28.** Two vessels contain mixtures of wine and water; in one there is three times as much wine as water, in the other five times as much water as wine. Find how much must be drawn off from each to fill a third vessel which holds seven gallons, in order that its contents may be half wine and half water.

**29.** There are two mixtures of wine and water, one of which contains twice as much water as wine, and the other three times as much wine as water. How much must there be taken from each to fill a pint cup, in which the water and the wine shall be equally mixed?

**30.** Two men set out at the same time to walk, one from $A$ to $B$, and the other from $B$ to $A$, a distance of $a$ miles. The former walks at the rate of $p$ miles, and the latter at the rate of $q$ miles an hour: at what distance from $A$ will they meet?

**31.** A train on the North Western line passes from London to Birmingham in 3 hours; a train on the Great Western line which is 15 miles longer, travelling at a speed which is less by 1 mile per hour, passes from one place to the other in $3\frac{1}{2}$ hours: find the length of each line.

**32.** Coffee is bought at 1*s.* and chicory at 3*d.* per lb.; in what proportion must they be mixed that 10 per cent. may be gained by selling the mixture at 11*d.* per lb.?

**33.** A man has one kind of coffee at $a$ pence per pound, and another at $b$ pence per pound. How much of each must he take to form a mixture of $a-b$ lbs., which he can sell at $c$ pence a pound without loss?

**34.** A man spends $c$ half-crowns in buying two kinds of silk at $a$ shillings and $b$ shillings a yard respectively; he could have bought 3 times as much of the first and half as much of the second for the same money: how many yards of each did he buy?

**35.** A man rides one-third of the distance from $A$ to $B$ at the rate of $a$ miles an hour, and the remainder at the rate of $2b$ miles an hour. If he had travelled at a uniform rate of $3c$ miles an hour, he could have ridden from $A$ to $B$ and back again in the same time. Prove that

$$\frac{2}{c}=\frac{1}{a}+\frac{1}{b}.$$

**36.** $A$, $B$, $C$ are three towns forming a triangle. A man has to walk from one to the next, ride thence to the next, and drive thence to his starting point. He can walk, ride, and drive a mile in $a$, $b$, $c$ minutes respectively. If he starts from $B$ he takes $a+c-b$ hours, if he starts from $C$ he takes $b+a-c$ hours, and if he starts from $A$ he takes $c+b-a$ hours. Find the length of the circuit.

# CHAPTER XXV.

## QUADRATIC EQUATIONS.

189. SUPPOSE the following problem were proposed for solution:

A dealer bought a number of horses for £280. If he had bought four less each would have cost £8 more: how many did he buy?

We should proceed thus:

Let $x$ = the number of horses; then $\frac{280}{x}$ = the number of pounds each cost.

If he had bought 4 less he would have had $x-4$ horses, and each would have cost $\frac{280}{x-4}$ pounds.

$$\therefore\ 8+\frac{280}{x}=\frac{280}{x-4};$$

whence

$$x(x-4)+35(x-4)=35x;$$

$$\therefore\ x^2-4x+35x-140=35x;$$

$$\therefore\ x^2-4x=140.$$

Here we have an equation which involves the *square* of the unknown quantity; and in order to complete the solution of the problem we must discover a method of solving such equations.

190. DEFINITION. An equation which contains the square of the unknown quantity, but no higher power, is called a **quadratic equation,** or an **equation of the second degree.**

If the equation contains both the square and the first power of the unknown it is called an *adfected* quadratic; if it contains only the square of the unknown it is said to be a *pure* quadratic.

Thus $2x^2-5x=3$ is an adfected quadratic,

and $5x^2=20$ is a pure quadratic.

191. A pure quadratic may be considered as a simple equation in which the *square* of the unknown quantity is to be found.

*Example.* Solve $\frac{9}{x^2-27}=\frac{25}{x^2-11}$.

Multiplying up, $9x^2-99=25x^2-675$;

$\therefore\ 16x^2=576$;

$\therefore\ x^2=36$;

and taking the square root of these equals, we have

$$x=\pm 6.$$

NOTE. We prefix the double sign to the number on the right-hand side for the reason given in Art. 117.

192. In extracting the square root of the two sides of the equation $x^2=36$, it might seem that we ought to prefix the double sign to the quantities on both sides, and write $\pm x=\pm 6$. But an examination of the various cases shews this to be unnecessary. For $\pm x=\pm 6$ gives the four cases:

$$+x=+6,\ \ +x=-6,\ \ -x=+6,\ \ -x=-6,$$

and these are all included in the two already given, namely $x=+6$, $x=-6$. Hence, when we extract the square root of the two sides of an equation, it is sufficient to put the double sign before the square root of *one* side.

193. The equation $x^2=36$ is an instance of the simplest form of quadratic equations. The equation $(x-3)^2=25$ may be solved in a similar way; for taking the square root of both sides, we have two *simple* equations,

$$x-3=\pm 5.$$

Taking the upper sign, $x-3=+5$, whence $x=8$;

taking the lower sign, $x-3=-5$, whence $x=-2$.

$\therefore$ the solution is $x=8$, or $-2$.

Now the given equation $(x-3)^2=25$

may be written $x^2-6x+(3)^2=25$,

or $x^2-6x=16$.

Hence, by retracing our steps, we learn that the equation

$$x^2-6x=16$$

can be solved by first adding $(3)^2$ or 9 to each side, and then extracting the square root; and the reason why we add 9 to each side is that this quantity added to the left side makes it a *perfect square.*

Now whatever the quantity $a$ may be,

$$x^2+2ax+a^2=(x+a)^2,$$

and

$$x^2-2ax+a^2=(x-a)^2;$$

so that if a trinomial is a perfect square, and *its highest power*, $x^2$, *has unity for its coefficient*, we must always have the term without $x$ equal to the square of half the coefficient of $x$. If, therefore, the terms in $x^2$ and $x$ are given, the square may be completed by adding the square of half the coefficient of $x$.

NOTE. When an expression is a perfect square, the *square terms* are always *positive*. [Art. 114, Note.] Hence, if necessary, the coefficient of $x^2$ must be made equal to $+1$ before completing the square.

*Example* 1. Solve $x^2+14x=32$.

The square of half 14 is $(7)^2$.

$$\therefore\ x^2+14x+(7)^2=32+49;$$

that is,

$$(x+7)^2=81;$$

$$\therefore\ x+7=\pm 9;$$

$$\therefore\ x=-7+9, \text{ or } -7-9;$$

$$\therefore\ x=2, \text{ or } -16.$$

*Example* 2. Solve $7x=x^2-8$.

Transpose so as to have the terms involving $x$ on one side, and the square term positive.

Thus

$$x^2-7x=8.$$

Completing the square, $x^2-7x+\left(\frac{7}{2}\right)^2=8+\frac{49}{4};$

that is,

$$\left(x-\frac{7}{2}\right)^2=\frac{81}{4};$$

$$\therefore\ x-\frac{7}{2}=\pm\frac{9}{2};$$

$$\therefore\ x=\frac{7}{2}\pm\frac{9}{2};$$

$$\therefore\ x=8, \text{ or } -1.$$

NOTE. We do not work out $\left(\frac{7}{2}\right)^2$ on the left-hand side.

## EXAMPLES XXV. a.

1. $5(x^2+5)=6x^2$.
2. $3x^2=4(x^2-4)$.
3. $x^2+22x=75$.
4. $x^2+24x=25$.
5. $x^2=10x-21$.
6. $(9+x)(9-x)=17$.
7. $x^2+3x=18$.
8. $x^2+5x=14$.
9. $x^2-5x-36=0$.

**10.** $x^2=x+72$.

**11.** $x^2-341=20x$.

**12.** $9x-x^2+220=0$.

**13.** $68-x^2=13x$.

**14.** $x+156=x^2$.

**15.** $187=x^2+6x$.

**16.** $23x=120+x^2$.

**17.** $42+x^2=13x$.

**18.** $22x+23-x^2=0$.

**19.** $x^2-\frac{2}{3}x=32$.

**20.** $x^2+\frac{4}{15}x=\frac{1}{5}$.

**21.** $x^2-\frac{7}{6}x-\frac{1}{2}=0$.

**22.** $\frac{19}{5}x=\frac{4}{5}-x^2$.

**23.** $\frac{3}{5}(x+6)(x-2)=\frac{2}{3}\left(62\frac{1}{10}+\frac{18x}{5}\right)$.

194. We have shewn that the square may readily be completed when the coefficient of $x^2$ is unity. All cases may be reduced to this by dividing the equation throughout by the coefficient of $x^2$.

*Example* 1. Solve $32-3x^2=10x$.

Transposing, $3x^2+10x=32$.

Divide throughout by 3, so as to make the coefficient of $x^2$ unity.

Thus
$$x^2+\frac{10}{3}x=\frac{32}{3},$$

completing the square,
$$x^2+\frac{10}{3}x+\left(\frac{5}{3}\right)^2=\frac{32}{3}+\frac{25}{9};$$

that is,
$$\left(x+\frac{5}{3}\right)^2=\frac{121}{9};$$

$$\therefore\ x+\frac{5}{3}=\pm\frac{11}{3};$$

$$\therefore\ x=-\frac{5}{3}\pm\frac{11}{3}=2, \text{ or } -5\frac{1}{3}.$$

Note. We do not add $\left(\frac{10}{6}\right)^2$ but $\left(\frac{5}{3}\right)^2$ to the left-hand side.

*Example* 2. Solve $5x^2+11x=12$.

Dividing by 5,
$$x^2+\frac{11}{5}x=\frac{12}{5};$$

completing the square,
$$x^2+\frac{11}{5}x+\left(\frac{11}{10}\right)^2=\frac{12}{5}+\frac{121}{100};$$

that is,
$$\left(x+\frac{11}{10}\right)^2=\frac{361}{100};$$

$$\therefore\ x+\frac{11}{10}=\pm\frac{19}{10};$$

$$\therefore\ x=-\frac{11}{10}\pm\frac{19}{10}=\frac{4}{5}, \text{ or } -3.$$

**195.** We see then that the following are the steps required for solving an adfected quadratic equation:

(1) If necessary, simplify the equation so that the terms in $x^2$ and $x$ are on one side of the equation, and the term without $x$ on the other.

(2) Make the coefficient of $x^2$ unity and positive by dividing throughout by the coefficient of $x^2$.

(3) Add to each side of the equation the square of half the coefficient of $x$.

(4) Take the square root of each side.

(5) Solve the resulting simple equations.

**196.** In the examples which follow some preliminary reduction and simplification may be necessary.

*Example* 1. Solve $\dfrac{3x-2}{2x-3}=\dfrac{5x}{x+4}-2.$

Simplifying, $\dfrac{3x-2}{2x-3}=\dfrac{3x-8}{x+4};$

multiplying across, $3x^2+10x-8=6x^2-25x+24;$

that is, $-3x^2+35x=32.$

Dividing by $-3$, $x^2-\dfrac{35}{3}x=-\dfrac{32}{3};$

completing the square, $x^2-\dfrac{35}{3}x+\left(\dfrac{35}{6}\right)^2=\dfrac{1225}{36}-\dfrac{32}{3};$

that is, $\left(x-\dfrac{35}{6}\right)^2=\dfrac{841}{36};$

$$\therefore\ x-\frac{35}{6}=\pm\frac{29}{6};$$

$$\therefore\ x=10\tfrac{2}{3},\ \text{or}\ 1.$$

*Example* 2. Solve $7(x+2a)^2+3a^2=5a(7x+23a).$

Simplifying, $7x^2+28ax+28a^2+3a^2=35ax+115a^2,$

that is, $7x^2-7ax=84a^2.$

Whence $x^2-ax=12a^2;$

completing the square, $x^2-ax+\left(\dfrac{a}{2}\right)^2=12a^2+\dfrac{a^2}{4};$

that is, $\left(x-\dfrac{a}{2}\right)^2=\dfrac{49a^2}{4};$

$$\therefore\ x-\frac{a}{2}=\pm\frac{7a}{2};$$

$$\therefore\ x=4a,\ \text{or}\ -3a.$$

197. In all the instances considered hitherto the quadratic equations have had two roots. Sometimes, however, there is only one solution. Thus if $x^2-2x+1=0$, then $(x-1)^2=0$, whence $x=1$ is the only solution. Nevertheless, in this and similar cases we find it convenient to say that the quadratic has *two equal roots.*

## EXAMPLES XXV. b.

**1.** $5x^2+14x=55.$ **2.** $3x^2+121=44x.$ **3.** $25x=6x^2+21.$

**4.** $8x^2+x=30.$ **5.** $3x^2+35=22x.$ **6.** $x+22-6x^2=0.$

**7.** $15=17x+4x^2.$ **8.** $21+x=2x^2.$ **9.** $9x^2-143-6x=0.$

**10.** $12x^2=29x-14.$ **11.** $20x^2=12-x.$

**12.** $19x=15-8x^2.$ **13.** $21x^2+22x+5=0.$

**14.** $50x^2-15x=27.$ **15.** $18x^2-27x-26=0.$

**16.** $5x^2=8x+21.$ **17.** $15x^2-2ax=a^2.$

**18.** $21x^2=2ax+3a^2.$ **19.** $6x^2=11kx+7k^2.$

**20.** $12x^2+23kx+10k^2=0.$ **21.** $12x^2-cx-20c^2=0.$

**22.** $2(x-3)=3(x+2)(x-3).$ **23.** $(x+1)(2x+3)=4x^2-22.$

**24.** $(3x-5)(2x-5)=x^2+2x-3.$

**25.** $\dfrac{5x+7}{x-1}=3x+2.$ **26.** $\dfrac{5x-1}{x+1}=\dfrac{3x}{2}.$ **27.** $\dfrac{3x-8}{x-2}=\dfrac{5x-2}{x+5}.$

**28.** $\dfrac{3x-1}{4x+7}=1-\dfrac{6}{x+7}.$ **29.** $\dfrac{5x-7}{7x-5}=\dfrac{x-5}{2x-13}.$

**30.** $\dfrac{x+3}{2x-7}-\dfrac{2x-1}{x-3}=0.$ **31.** $\dfrac{1}{1+x}-\dfrac{1}{3-x}=\dfrac{6}{35}.$

**32.** $\dfrac{x+4}{x-4}+\dfrac{x-2}{x-3}=6\tfrac{1}{3}.$ **33.** $\dfrac{1}{3-x}-\dfrac{4}{5}=\dfrac{1}{9-2x}.$

**34.** $\dfrac{4}{x-1}-\dfrac{5}{x+2}=\dfrac{3}{x}.$ **35.** $\dfrac{5}{x-2}-\dfrac{4}{x}=\dfrac{3}{x+6}.$

198. From the preceding examples it appears that after suitable reduction and transposition every quadratic equation can be written in the form

$$ax^2+bx+c=0,$$

where $a$, $b$, $c$ may have any numerical values whatever. If therefore we can solve this quadratic we can solve any.

Transposing, $ax^2+bx=-c;$

dividing by $a$, $x^2+\frac{b}{a}x=-\frac{c}{a}.$

Completing the square by adding to each side $\left(\frac{b}{2a}\right)^2$,

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a};$$

that is, $$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2};$$

extracting the square root,

$$x+\frac{b}{2a}=\frac{\pm\sqrt{(b^2-4ac)}}{2a};$$

$$\therefore\ x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}.$$

199. Instead of going through the process of completing the square in each particular example, we may now make use of this general formula, adapting it to the case in question by substituting the values of $a$, $b$, $c$.

*Example.* Solve $5x^2+11x-12=0.$

Here $a=5$, $b=11$, $c=-12$.

$$\therefore\ x=\frac{-11\pm\sqrt{(11)^2-4\,.\,5\,(-12)}}{10}$$

$$=\frac{-11\pm\sqrt{361}}{10}=\frac{-11\pm19}{10}=\frac{4}{5},\ \text{or}\ -3,$$

which agrees with the solution of Example 2, Art. 194.

200. In the result $x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$,

it must be remembered that the expression $\sqrt{(b^2-4ac)}$ is the square root of the compound quantity $b^2-4ac$, *taken as a whole.* We cannot simplify the solution unless we know the numerical values of $a$, $b$, $c$. It may sometimes happen that these values do not make $b^2-4ac$ a perfect square. In such a case the exact numerical solution of the equation cannot be determined.

*Example* 1. Solve $5x^2-15x+11=0$.

We have
$$x=\frac{15\pm\sqrt{(-15)^2-4.5.11}}{2.5}$$
$$=\frac{15\pm\sqrt{5}}{10}.$$

Now $\sqrt{5}=2{\cdot}236$ approximately.
$$\therefore\ x=\frac{15\pm2{\cdot}236}{10}=1{\cdot}7236, \text{ or } 1{\cdot}2764.$$

These solutions are correct only to four places of decimals, and neither of them will be found to *exactly* satisfy the equation.

Unless the *numerical* values of the unknown quantity are required it is usual to leave the roots in the form
$$\frac{15+\sqrt{5}}{10},\quad \frac{15-\sqrt{5}}{10}.$$

*Example* 2. Solve $x^2-3x+5=0$.

We have
$$x=\frac{3\pm\sqrt{(-3)^2-4.1.5}}{2}$$
$$=\frac{3\pm\sqrt{9-20}}{2}$$
$$=\frac{3\pm\sqrt{-11}}{2}.$$

But $-11$ has no square root exact or approximate [Art. 117]; so that no real value of $x$ can be found to satisfy the equation. In such a case the roots are said to be *imaginary* or *impossible*.

201. There is still one method of obtaining the solution of a quadratic which will sometimes be found shorter than either of the methods already given.

Consider the equation $x^2+\frac{7}{3}x=2$.

Clearing of fractions, $3x^2+7x-6=0$ ....................(1);
by resolving the left-hand side into factors we have
$$(3x-2)(x+3)=0.$$
Now if *either* of the factors $3x-2$, $x+3$ be zero their product is zero. Hence the quadratic equation is satisfied by either of the suppositions
$$3x-2=0, \text{ or } x+3=0.$$
Thus the roots are $\frac{2}{3}$, $-3$.

It appears from this that *when a quadratic equation has been simplified and brought to the form of equation* (1), its solution can always be readily obtained if the expression on the left-hand side can be resolved into factors. Each of these factors equated to zero gives a simple equation, and a corresponding root of the quadratic.

*Example* 1. Solve $2x^2 - ax + 2bx = ab$.

Transposing, *so as to have all the terms on one side of the equation,* we have

$$2x^2 - ax + 2bx - ab = 0.$$

Now $2x^2 - ax + 2bx - ab = x(2x - a) + b(2x - a)$

$$= (2x - a)(x + b).$$

Therefore $(2x - a)(x + b) = 0$;

whence $2x - a = 0$, or $x + b = 0$.

$$\therefore\ x = \frac{a}{2}, \text{ or } -b.$$

*Example* 2. Solve $2(x^2 - 6) = 3(x - 4)$.

We have $2x^2 - 12 = 3x - 12$;

that is, $2x^2 = 3x$ ................................ (1).

Transposing, $2x^2 - 3x = 0$,

$$x(2x - 3) = 0.$$

$$\therefore\ x = 0, \text{ or } 2x - 3 = 0.$$

Thus the roots are 0, $\frac{3}{2}$.

NOTE. In equation (1) above we might have divided both sides by $x$ and obtained the simple equation $2x = 3$, whence $x = \frac{3}{2}$, which is *one* of the solutions of the given equation. But the student must be particularly careful to notice that whenever an $x$ is removed by division from every term of an equation it must not be neglected, since the equation is satisfied by $x = 0$, which is therefore one of the roots.

202. There are some equations which are not really quadratics, but which may be solved by the methods explained in this chapter.

*Example* 1. Solve $x^4 - 13x^2 + 36 = 0$.

By resolution into factors, $(x^2 - 9)(x^2 - 4) = 0$;

$$\therefore\ x^2 - 9 = 0, \text{ or } x^2 - 4 = 0;$$

that is, $x^2 = 9$, or 4,

and $x = \pm 3$, or $\pm 2$.

*Example* 2. Solve $x^2+3x-\dfrac{20}{x^2+3x}=8$.

Write $y$ for $x^2+3x$, then we have

$$y-\frac{20}{y}=8,$$

or
$$y^2-8y-20=0.$$

From this quadratic $\quad y=10, \text{ or } -2;$

$$\therefore\ x^2+3x=10, \text{ or } -2.$$

Thus we have *two* quadratics to solve, and finally we obtain $x=-5, 2$; or $-1, -2$.

## EXAMPLES XXV. c.

Solve by the aid of the formula in Art. 198:

**1.** $3x^2=15-4x$. **2.** $2x^2+7x=15$. **3.** $2x^2+7-9x=0$.

**4.** $x^2=3x+5$. **5.** $5x^2+4+21x=0$. **6.** $x^2+11=7x$.

**7.** $8x^2=x+7$. **8.** $5x^2=17x-10$.

**9.** $35+9x-2x^2=0$. **10.** $3x^2=x+1$.

**11.** $3x^2+5x=2$. **12.** $2x^2+5x-33=0$.

Solve by resolution into factors:

**13.** $6x^2=7+x$. **14.** $21+8x^2=26x$.

**15.** $26x-21+11x^2=0$. **16.** $5x^2+26x+24=0$.

**17.** $4x^2=\dfrac{4}{15}x+3$. **18.** $x^2-2=\frac{23}{12}x$.

**19.** $7x^2=28-96x$. **20.** $96x^2=4x+15$.

**21.** $25x^2=5x+6$. **22.** $35-4x=4x^2$.

**23.** $12x^2-11ax=36a^2$. **24.** $12x^2+36a^2=43ax$.

**25.** $35b^2=9x^2+6bx$. **26.** $36x^2-35b^2=12bx$.

**27.** $x^2-2ax+4ab=2bx$. **28.** $x^2-2ax+8x=16a$.

**29.** $3x^2-2ax-bx=0$. **30.** $ax^2+2x=bx$.

Solve as explained in Art. 202:

**31.** $4=5x^2-x^4$. **32.** $x^4+36=13x^2$.

**33.** $x^6+7x^3=8$. **34.** $x^6-19x^3=216$.

**35.** $16\left(x^2+\dfrac{1}{x^2}\right)=257$. **36.** $x^2+\dfrac{a^2b^2}{x^2}=a^2+b^2$.

**37.** $x^3(19+x^3)=216$. **38.** $(x^2+2)^2+198=29(x^2+2)$.

**39.** $x^2-x+\dfrac{72}{x^2-x}=18$. **40.** $x(x-2a)=\dfrac{8a^4}{x^2-2ax}+7a^2$.

# CHAPTER XXVI.

## SIMULTANEOUS QUADRATIC EQUATIONS.

203. WE shall now consider some of the most useful methods of solving simultaneous equations, one or more of which may be of a degree higher than the first; but no fixed rules can be laid down which are applicable to all cases.

*Example* 1. Solve $x+y=15 \qquad\qquad (1),$
$xy=36 \qquad\qquad (2).$

From (1) by squaring, $x^2+2xy+y^2=225$;
from (2) $4xy=144$;
by subtraction, $x^2-2xy+y^2=81$;
by taking the square root, $x-y=\pm 9$.

Combining this with (1) we have to consider the two cases,

$$\left.\begin{array}{l} x+y=15, \\ x-y=\ \ 9. \end{array}\right\} \qquad \left.\begin{array}{l} x+y=\ \ 15, \\ x-y=-\ 9. \end{array}\right\}$$

from which we find

$$\left.\begin{array}{l} x=12, \\ y=\ \ 3. \end{array}\right\} \qquad \left.\begin{array}{l} x=\ \ 3, \\ y=12. \end{array}\right\}$$

*Example* 2. Solve $x-y=12 \qquad\qquad (1),$
$xy=85 \qquad\qquad (2).$

From (1) $x^2-2xy+y^2=144$;
from (2) $4xy=340$;
by addition, $x^2+2xy+y^2=484$;
by taking the square root, $x+y=\pm 22$.

Combining this with (1) we have the two cases,

$$\left.\begin{array}{l} x+y=22, \\ x-y=12. \end{array}\right\} \qquad \left.\begin{array}{l} x+y=-\ 22, \\ x-y=\ \ 12. \end{array}\right\}$$

Whence

$$\left.\begin{array}{l} x=17, \\ y=\ \ 5. \end{array}\right\} \qquad \left.\begin{array}{l} x=-\ 5, \\ y=-17. \end{array}\right\}$$

204. These are the simplest cases that arise, but they are specially important since the solution in a large number of other cases is dependent upon them.

As a rule our object is to solve the proposed equations *symmetrically*, by finding the values of $x+y$ and $x-y$. From the foregoing examples it will be seen that we can always do this as soon as we have obtained the product of the unknowns, and either their sum or their difference.

*Example* 1. Solve

$$x^2+y^2=74 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$xy=35 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Multiply (2) by 2; then by addition and subtraction we have

$$x^2+2xy+y^2=144,$$
$$x^2-2xy+y^2=4;$$

Whence

$$x+y=\pm 12,$$
$$x-y=\pm 2.$$

We have now four cases to consider; namely,

$$\left.\begin{array}{l} x+y=12, \\ x-y=2. \end{array}\right\} \qquad \left.\begin{array}{l} x+y=12, \\ x-y=-2. \end{array}\right\} \qquad \left.\begin{array}{l} x+y=-12, \\ x-y=2. \end{array}\right\} \qquad \left.\begin{array}{l} x+y=-12, \\ x-y=-2. \end{array}\right\}$$

From which the values of $x$ are $7, 5, -5, -7$; and the corresponding values of $y$ are $5, 7, -7, -5$.

*Example* 2. Solve

$$x^2+y^2=185 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$x+y=17 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

By subtracting (1) from the square of (2) we have

$$2xy=104;$$
$$\therefore\ xy=52 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Equations (2) and (3) can now be solved by the method of Art. 203, Example 1; and the solution is

$$\left.\begin{array}{l} x=13, \text{ or } 4, \\ y=4, \text{ or } 13. \end{array}\right\}$$

## EXAMPLES XXVI. a.

Solve the following equations:

1. $x+y=28$, $xy=187$.
2. $x+y=51$, $xy=518$.
3. $x+y=74$, $xy=1113$.
4. $x-y=5$, $xy=126$.
5. $x-y=8$, $xy=513$.
6. $xy=1075$, $x-y=18$.
7. $xy=923$, $x+y=84$.
8. $x-y=-8$, $xy=1353$.
9. $x-y=-22$, $xy=3848$.

10. $xy=-2193,\quad x+y=-8.$

11. $x-y=-18,\quad xy=1363.$

12. $xy=-1914,\quad x+y=-65.$

13. $x^2+y^2=89,\quad xy=40.$

14. $x^2+y^2=170,\quad xy=13.$

15. $x^2+y^2=74,\quad xy=35.$

16. $x^2+y^2=178,\quad x+y=16.$

17. $x+y=15,\quad x^2+y^2=125.$

18. $x-y=4,\quad x^2+y^2=106.$

19. $x^2+y^2=180,\quad x-y=6.$

20. $x^2+y^2=185,\quad x-y=3.$

21. $x+y=13,\quad x^2+y^2=97.$

22. $x+y=9,\quad x^2+xy+y^2=61.$

23. $x-y=3,\quad x^2-3xy+y^2=-19.$

24. $x^2-xy+y^2=76,\quad x+y=14.$

25. $\frac{1}{10}(x-y)=1,\quad x^2-4xy+y^2=52.$

26. $\frac{1}{x}+\frac{1}{y}=2,\quad x+y=2.$

27. $\frac{1}{x}+\frac{1}{y}=\frac{7}{12},\quad xy=12.$

28. $ax+by=2,\quad abxy=1.$

29. $x^2+pxy+y^2=p+2,\quad qx^2+xy+qy^2=2q+1.$

205. Any pair of equations of the form

$$x^2 \pm pxy+y^2=a^2 \quad\dots\dots\dots(1),$$
$$x \pm y=b \quad\dots\dots\dots(2),$$

where $p$ is any numerical quantity, can be reduced to one of the cases already considered; for by squaring (2) and combining with (1), an equation to find $xy$ is obtained; the solution can then be completed by the aid of equation (2).

*Example* 1. Solve $x^3-y^3=999$ ......... (1),

$x-y=\ \ 3$ ......... (2).

By division, $x^2+xy+y^2=333$ ......... (3);

from (2) $x^2-2xy+y^2=\ \ 9;$

by subtraction, $3xy=324,$

$xy=108$ ......... (4).

From (2) and (4) $\left.\begin{aligned}x&=12, \text{ or } -9,\\ y&=9, \text{ or } -12.\end{aligned}\right\}$

*Example* 2. Solve $x^4+x^2y^2+y^4=2613$ .........(1),

$x^2+xy+y^2=\ \ 67$ .........(2).

Dividing (1) by (2) $x^2-xy+y^2=\ \ 39$ .........(3).

From (2) and (3) by addition, $x^2+y^2=53;$

by subtraction, $xy=9;$

whence $\left.\begin{aligned}x&=\pm 7,\ \pm 2,\\ y&=\pm 2,\ \pm 7.\end{aligned}\right\}$ [Art. 204, Ex. 1.]

*Example* 3. Solve

$$\frac{1}{x}-\frac{1}{y}=\frac{1}{3}\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{9}\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) by squaring, $\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{9};$

by subtraction, $\frac{2}{xy}=\frac{4}{9};$

adding to (2), $\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=1;$

$$\therefore\ \frac{1}{x}+\frac{1}{y}=\pm 1.$$

Combining with (1), $\frac{1}{x}=\frac{2}{3},\text{ or }-\frac{1}{3},$

$$\frac{1}{y}=\frac{1}{3},\text{ or }-\frac{2}{3};$$

$$\therefore\ \left.\begin{aligned}x&=\frac{3}{2},\text{ or }-3,\\ y&=3,\text{ or }-\frac{3}{2}.\end{aligned}\right\}$$

## EXAMPLES XXVI. b.

1. $x^3+y^3=407,$ $x+y=11.$

2. $x^3+y^3=637,$ $x+y=13.$

3. $x+y=23,$ $x^3+y^3=3473.$

4. $x^3-y^3=218,$ $x-y=2.$

5. $x-y=4,$ $x^3-y^3=988.$

6. $x^3-y^3=2197,$ $x-y=13.$

7. $x^4+x^2y^2+y^4=2128,$ $x^2+xy+y^2=76.$

8. $x^4+x^2y^2+y^4=2923,$ $x^2-xy+y^2=37.$

9. $x^4+x^2y^2+y^4=9211,$ $x^2-xy+y^2=61.$

10. $x^4+x^2y^2+y^4=7371,$ $x^2-xy+y^2=63.$

11. $\frac{1}{x^2}+\frac{1}{y^2}=\frac{481}{576},$ $\frac{1}{x}+\frac{1}{y}=\frac{29}{24}.$

12. $\frac{1}{x^2}+\frac{1}{y^2}=\frac{61}{900},$ $xy=30.$

13. $\frac{x}{y}+\frac{y}{x}=2\frac{1}{2},$ $x+y=6.$

14. $\frac{x}{y}+\frac{y}{x}=2\frac{16}{21},$ $x-y=4.$

15. $\frac{34}{x^2+y^2}=\frac{15}{xy},$ $x+y=8.$

16. $x^3-y^3=56,$ $x^2+xy+y^2=28.$

17. $4(x^2+y^2)=17xy,$ $x-y=6.$

18. $x^3+y^3=126,$ $x^2-xy+y^2=21.$

19. $\frac{1}{x^3}+\frac{1}{y^3}=1\frac{1}{125}$,
$\frac{1}{x}+\frac{1}{y}=1\frac{1}{5}$.

20. $\frac{1}{x^3}-\frac{1}{y^3}=91$,
$\frac{1}{x}-\frac{1}{y}=1$.

206. The following method of solution may always be used when the equations are *of the same degree and homogeneous.* [See Art. 12.]

*Example.* Solve

$$x^2+xy+2y^2=74 \quad\text{.........................}(1),$$

$$2x^2+2xy+y^2=73 \quad\text{.........................}(2).$$

Put $y=mx$, and substitute in both equations. Thus

$$x^2(1+m+2m^2)=74 \quad\text{.........................}(3),$$

and

$$x^2(2+2m+m^2)=73 \quad\text{.........................}(4).$$

By division,

$$\frac{1+m+2m^2}{2+2m+m^2}=\frac{74}{73};$$

$$\therefore\ 73+73m+146m^2=148+148m+74m^2;$$

$$\therefore\ 72m^2-75m-75=0,$$

or

$$24m^2-25m-25=0;$$

$$\therefore\ (8m+5)(3m-5)=0;$$

$$\therefore\ m=-\frac{5}{8},\ \text{or}\ \frac{5}{3}.$$

(i) Take $m=-\frac{5}{8}$, and substitute in either (3) or (4).

From (3)

$$x^2\left(1-\frac{5}{8}+\frac{50}{64}\right)=74;$$

$$\therefore\ x^2=\frac{64\times 74}{74}=64;$$

$$\therefore\ x=\pm 8;$$

$$\therefore\ y=mx=-\frac{5}{8}x=\mp 5.$$

(ii) Take $m=\frac{5}{3}$; then from (3)

$$x^2\left(1+\frac{5}{3}+\frac{50}{9}\right)=74,$$

$$x^2=\frac{74\times 9}{74}=9;$$

$$\therefore\ x=\pm 3;$$

$$\therefore\ y=mx=\frac{5}{3}x=\pm 5.$$

207. When one of the equations is of the first degree and the other of a higher degree, we may from the simple equation find the value of one of the unknowns in terms of the other, and substitute in the second equation.

*Example.* Solve $3x-4y=5$ .............................. (1),

$3x^2-xy-3y^2=21$ .............................. (2).

From (1) we have $x=\dfrac{5+4y}{3}$;

and substituting in (2), $\dfrac{3(5+4y)^2}{9}-\dfrac{y(5+4y)}{3}-3y^2=21$;

$$\therefore\ 75+120y+48y^2-15y-12y^2-27y^2=189;$$
$$9y^2+105y-114=0,$$
$$3y^2+35y-38=0;$$
$$\therefore\ (y-1)(3y+38)=0;$$
$$\therefore\ y=1, \text{ or } -\frac{38}{3};$$

and by substituting in (1), $x=3$, or $-\dfrac{137}{9}$.

208. The examples we have given will be sufficient as a general explanation of the methods to be employed; but in some cases special artifices are necessary.

*Example* 1. Solve $x^2+4xy+3x=40-6y-4y^2$..................(1),

$2xy-x^2=3$ .............................. (2).

From (1) we have $x^2+4xy+4y^2+3x+6y=40$;

that is, $(x+2y)^2+3(x+2y)-40=0$,

or $(\overline{x+2y}+8)(\overline{x+2y}-5)=0$;

whence $x+2y=-8$, or $5$.

(i) Combining $x+2y=5$ with (2) we obtain

$$2x^2-5x+3=0;$$

whence $x=1$, or $\dfrac{3}{2}$;

and by substituting in $x+2y=5$, $y=2$, or $\dfrac{7}{4}$.

(ii) Combining $x+2y=-8$ with (2) we obtain

$$2x^2+8x+3=0;$$

whence $x=\dfrac{-4\pm\sqrt{10}}{2}$; and $y=\dfrac{-12\mp\sqrt{10}}{4}$.

*Example* 2. Solve $x^2y^2 - 6x = 34 - 3y$ ........................ (1),

$3xy + y = 2(9 + x)$ ........................ (2).

From (1) $x^2y^2 - 6x + 3y = 34;$

from (2) $9xy - 6x + 3y = 54;$

by subtraction, $x^2y^2 - 9xy + 20 = 0,$

$(xy - 5)(xy - 4) = 0;$

$\therefore\ xy = 5$, or 4.

(i) Substituting $xy = 5$ in (2) gives $y - 2x = 3$.

From these equations we obtain $\left.\begin{aligned} x &= 1, \text{ or } -\frac{5}{2}, \\ y &= 5, \text{ or } -2. \end{aligned}\right\}$

(ii) Substituting $xy = 4$ in (2) gives $y - 2x = 6$.

From these equations we obtain $x = \dfrac{-3 \pm \sqrt{17}}{2}$,

and $y = 3 \pm \sqrt{17}$.

## EXAMPLES XXVI. c.

1. $5x - y = 17,\quad xy = 12.$

2. $x^2 + xy = 15,\quad y^2 + xy = 10.$

3. $x - y = 10,\quad x^2 - 2xy - 3y^2 = 84.$

4. $3x + 2y = 16,\quad xy = 10.$

5. $3x - y = 11,\quad 3x^2 - y^2 = 47.$

6. $x - 3y = 1,\quad x^2 - 2xy + 9y^2 = 17.$

7. $x + 2y = 9,\quad 3y^2 - 5x^2 = 43.$

8. $x^2 - xy = 24,\quad xy - y^2 = 8.$

9. $5x + y = 3,\quad 2x^2 - 3xy - y^2 = 1.$

10. $x^2 + 4xy = 133,\quad 4xy + 16y^2 = 228.$

11. $2x^2 + 3xy = 26,\quad 3y^2 + 2xy = 39.$

12. $x^2 + xy + y^2 = 3\frac{1}{4},\quad 2x^2 - 3xy + 2y^2 = 2\frac{3}{4}.$

13. $x^2 - 3xy + y^2 + 1 = 0,\quad 3x^2 - xy + 3y^2 = 13.$

14. $2x^2 - 5xy + 3y^2 = 1,\quad 3x^2 - 5xy + 2y^2 = 4.$

15. $x^2 - 2xy = 21,\quad xy + y^2 = 18.$

16. $x^2 + 3xy = 54,\quad xy + 4y^2 = 115.$

17. $x^3 + y^3 = 152,\quad x^2y + xy^2 = 120.$

18. $x^3 - y^3 = 127,\quad x^2y - xy^2 = 42.$

19. $x^3 - y^3 = 208,\quad xy(x - y) = 48.$

20. $x^2y^2 + 5xy = 84,\quad x + y = 8.$

21. $x^2 + 4y^2 + 80 = 15x + 30y,\quad xy = 6.$

22. $9x^2 + y^2 - 63x - 21y + 128 = 0,\quad xy = 4.$

# CHAPTER XXVII.

## PROBLEMS LEADING TO QUADRATIC EQUATIONS.

209. WE shall now discuss some problems which give rise to quadratic equations.

*Example* 1. **A train travels 300 miles at a uniform rate; if the rate had been 5 miles an hour more, the journey would have taken two hours less: find the rate of the train.**

Suppose the train travels at the rate of $x$ miles per hour, then the time occupied is $\frac{300}{x}$ hours.

On the other supposition the time is $\frac{300}{x+5}$ hours;

$$\therefore \frac{300}{x+5} = \frac{300}{x} - 2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

whence $\qquad x^2 + 5x - 750 = 0,$

or $\qquad (x+30)(x-25) = 0,$

$$\therefore x = 25, \text{ or } -30.$$

Hence the train travels 25 miles per hour, the negative value being inadmissible.

It will frequently happen that the algebraical statement of the question leads to a result which does not apply to the actual problem we are discussing. But such results can sometimes be explained by a suitable modification of the conditions of the question. In the present case we may explain the negative solution as follows.

Since the values $x = 25$ and $-30$ satisfy the equation (1), if we write $-x$ for $x$ the resulting equation,

$$\frac{300}{-x+5} = \frac{300}{-x} - 2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

will be satisfied by the values $x = -25$ and 30. Now, by changing signs throughout, equation (2) becomes $\frac{300}{x-5} = \frac{300}{x} + 2$;

and this is the algebraical statement of the following question:

A train travels 300 miles at a uniform rate; if the rate had been 5 miles an hour *less*, the journey would have taken two hours *more*: find the rate of the train. The rate is 30 miles an hour.

*Example* 2. A person selling a horse for £72 finds that his loss per cent. is one-eighth of the number of pounds that he paid for the horse: what was the cost price?

Suppose that the cost price of the horse is $x$ pounds; then the loss on £100 is £$\frac{x}{8}$.

Hence the loss on £$x$ is $x \times \frac{x}{800}$, or $\frac{x^2}{800}$ pounds;

$\therefore$ the selling price is $x - \frac{x^2}{800}$ pounds.

Hence $$x - \frac{x^2}{800} = 72,$$

or $$x^2 - 800x + 57600 = 0;$$

that is, $$(x-80)(x-720) = 0;$$

$$\therefore x = 80, \text{ or } 720;$$

and each of these values will be found to satisfy the conditions of the problem. Thus the cost is either £80, or £720.

*Example* 3. A cistern can be filled by two pipes in $33\frac{1}{3}$ minutes; if the larger pipe takes 15 minutes less than the smaller to fill the cistern, find in what time it will be filled by each pipe singly.

Suppose that the two pipes running singly would fill the cistern in $x$ and $x-15$ minutes. When running together they will fill $\left(\frac{1}{x} + \frac{1}{x-15}\right)$ of the cistern in one minute. But they fill $\frac{1}{33\frac{1}{3}}$, or $\frac{3}{100}$ of the cistern in one minute;

Hence $$\frac{1}{x} + \frac{1}{x-15} = \frac{3}{100},$$

$$100(2x-15) = 3x(x-15),$$

$$3x^2 - 245x + 1500 = 0,$$

$$(x-75)(3x-20) = 0;$$

$$\therefore x = 75, \text{ or } 6\tfrac{2}{3}.$$

Thus the smaller pipe takes 75 minutes, the larger 60 minutes.

The other solution $6\frac{2}{3}$ is inadmissible.

*Example* 4. The small wheel of a bicycle makes 135 revolutions more than the large wheel in a distance of 260 yards; if the circumference of each were one foot more, the small wheel would make 27 revolutions more than the large wheel in a distance of 70 yards: find the circumference of each wheel.

Suppose the small wheel to be $x$ feet, and the large wheel $y$ feet in circumference.

In a distance of 260 yards the two wheels make $\frac{780}{x}$ and $\frac{780}{y}$ revolutions respectively.

Hence $$\frac{780}{x} - \frac{780}{y} = 135,$$

or $$\frac{1}{x} - \frac{1}{y} = \frac{9}{52} \quad \dots\dots\dots\dots\dots\dots (1).$$

Similarly from the second condition, we obtain

$$\frac{210}{x+1} - \frac{210}{y+1} = 27,$$

or $$\frac{1}{x+1} - \frac{1}{y+1} = \frac{9}{70} \quad \dots\dots\dots\dots\dots\dots (2).$$

From (1) $$x = \frac{52y}{52+9y};$$

whence $$x+1 = \frac{61y+52}{9y+52}.$$

Substituting in (2), $$\frac{9y+52}{61y+52} - \frac{1}{y+1} = \frac{9}{70},$$

$$70 \times 9y^2 = 9(61y+52)(y+1),$$

$$9y^2 - 113y - 52 = 0,$$

$$(y-13)(9y+4) = 0,$$

$$\therefore\ y = 13, \text{ or } -\frac{4}{9}.$$

Putting $y=13$ we find that $x=4$. The other value of $y$ is inadmissible; hence the small wheel is 4 feet, the large wheel 13 feet in circumference.

*Example* 5. On a river there are two towns 24 miles apart. By rowing one half of the distance and walking the other half, a man performs the journey down stream in 5 hours, and up stream in 7 hours. Had there been no current, each journey would have taken $5\frac{2}{3}$ hours: find the rate of his walking, and rowing, and the rate of the stream.

Suppose that the man walks $x$ miles per hour, rows $y$ miles per hour, and that the stream flows at the rate of $z$ miles per hour.

With the current the man rows $y+z$ miles, and against the current $y-z$ miles per hour.

Hence we have the following equations:

$$\frac{12}{x}+\frac{12}{y+z}=5 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{12}{x}+\frac{12}{y-z}=7 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\frac{12}{x}+\frac{12}{y}=5\tfrac{2}{3} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1) and (3) by subtraction, $\frac{1}{y}-\frac{1}{y+z}=\frac{1}{18} \ldots\ldots\ldots\ldots (4).$

Similarly, from (2) and (3) $\frac{1}{y-z}-\frac{1}{y}=\frac{1}{9} \ldots\ldots\ldots\ldots (5).$

From (4) $18z=y\,(y+z) \ldots\ldots\ldots\ldots\ldots (6);$

and from (5) $9z=y\,(y-z) \ldots\ldots\ldots\ldots\ldots (7).$

From (6) and (7) by division, $2=\frac{y+z}{y-z}$;

whence $y=3z$;

$\therefore$ from (4) $z=1\tfrac{1}{2}$; and hence $y=4\tfrac{1}{2}$, $x=4$.

Thus the rates of walking and rowing are 4 miles and $4\tfrac{1}{2}$ miles per hour respectively; and the stream flows at the rate of $1\tfrac{1}{2}$ miles per hour.

## EXAMPLES XXVII.

1. Find a number whose square diminished by 119 is equal to ten times the excess of the number over 8.

2. A man is five times as old as his son, and the sum of the squares of their ages is equal to 2106: find their ages.

3. The sum of the reciprocals of two consecutive numbers is $\frac{15}{56}$; find them.

4. Find a number which when increased by 17 is equal to 60 times the reciprocal of the number.

5. Find two numbers whose sum is 9 times their difference, and the difference of whose squares is 81.

6. The sum of a number and its square is nine times the next highest number; find it.

7. If a train travelled 5 miles an hour faster it would take one hour less to travel 210 miles: what time does it take?

8. Find two numbers the sum of whose squares is 74, and whose sum is 12.

9. The perimeter of a rectangular field is 500 yards, and its area is 14400 square yards: find the length of the sides.

10. The perimeter of one square exceeds that of another by 100 feet; and the area of the larger square exceeds three times the area of the smaller by 325 square feet: find the length of their sides.

11. A cistern can be filled by two pipes running together in 22½ minutes: the larger pipe would fill the cistern in 24 minutes less than the smaller one: find the time taken by each.

12. A man travels 108 miles, and finds that he could have made the journey in 4½ hours less had he travelled 2 miles an hour faster: at what rate did he travel?

13. I buy a number of cricket balls for £5; had they cost a shilling apiece less, I should have had five more for the money: find the cost of each.

14. A boy was sent out for a shilling's worth of eggs. He broke 3 on his way home, and his master therefore had to pay at the rate of a penny more than the market price for 5. How many did the master get for a shilling?

15. What are eggs a dozen when two more in a shilling's worth lowers the price a penny per dozen?

16. A lawn 50 feet long and 34 feet broad has a path of uniform width round it; if the area of the path is 540 square feet, find its width.

17. A hall can be paved with 200 square tiles of a certain size; if each tile were one inch longer each way it would take 128 tiles: find the length of each tile.

18. In the centre of a square garden is a square lawn; outside this is a gravel walk 4 feet wide, and then a flower border 6 feet wide. If the flower border and lawn together contain 721 square feet, find the area of the lawn.

19. By lowering the price of apples and selling them one penny a dozen cheaper, an applewoman finds that she can sell 60 more than she used to do for 5*s*. At what price per dozen did she sell them at first?

20. Two rectangles contain the same area, 480 square yards. The difference of their lengths is 10 yards, and of their breadths 4 yards; find their sides.

21. There is a number between 10 and 100; when multiplied by the digit on the left the product is 280; if the sum of the digits be multiplied by the same digit the product is 55: required the number.

22. A farmer having sold at 75*s.* a head, a flock of sheep which cost him $x$ shillings a head, finds that he has realised $x$ per cent. profit on his outlay; find $x$.

23. A tradesman bought a number of yards of cloth for £5; he kept 5 yards and sold the rest at 2*s.* per yard more than he gave, and got £1 more than he originally spent: how many yards did he buy?

24. If a carriage wheel $14\frac{2}{3}$ ft. in circumference takes one second more to revolve, the rate of the carriage per hour will be $2\frac{2}{3}$ miles less: how fast is the carriage travelling?

25. A broker bought as many railway shares as cost him £1875; he reserved 15, and sold the remainder for £1740, gaining £4 a share on their cost price. How many shares did he buy?

26. $A$ and $B$ are two stations 300 miles apart. Two trains start simultaneously from $A$ and $B$, each to the opposite station. The train from $A$ reaches $B$ nine hours, the train from $B$ reaches $A$ four hours after they meet: find the rate at which each train travels.

27. A train $A$ starts to go from $P$ to $Q$, two stations 240 miles apart, and travels uniformly. An hour later another train $B$ starts from $P$, and after travelling for 2 hours, comes to a point that $A$ had passed 45 minutes previously. The pace of $B$ is now increased by 5 miles an hour, and it overtakes $A$ just on entering $Q$. Find the rates at which they started.

28. A cask $P$ is filled with 50 gallons of water, and a cask $Q$ with 40 gallons of brandy; $x$ gallons are drawn from each cask, mixed and replaced; and the same operation is repeated. Find $x$ when there are $8\frac{7}{8}$ gallons of brandy in $P$ after the second replacement.

29. Two farmers $A$ and $B$ have 30 cows between them; they sell at different prices, but each receives the same sum. If $A$ had sold his at $B$'s price, he would have received £320; and if $B$ had sold his at $A$'s price, he would have received £245. How many had each?

30. A man arrives at the railway station nearest to his house $1\frac{1}{2}$ hours before the time at which he had ordered his carriage to meet him. He sets out at once to walk at the rate of 4 miles an hour, and, meeting his carriage when it had travelled 8 miles, reaches home exactly 1 hour earlier than he had originally expected. How far is his house from the station, and at what rate was his carriage driven?

# CHAPTER XXVIII.

## HARDER FACTORS.

210. IN Chapter XVII. we have explained several rules for resolving algebraical expressions into factors; in the present Chapter we shall continue the subject by discussing cases of greater difficulty.

211. By a slight modification some expressions admit of being written in the form of the difference of two squares, and may then be resolved into factors by the method of Art. 133.

*Example* 1. Resolve into factors $x^4+x^2y^2+y^4$.

$$\begin{aligned}x^4+x^2y^2+y^4&=(x^4+2x^2y^2+y^4)-x^2y^2\\&=(x^2+y^2)^2-(xy)^2\\&=(x^2+y^2+xy)(x^2+y^2-xy)\\&=(x^2+xy+y^2)(x^2-xy+y^2).\end{aligned}$$

*Example* 2. Resolve into factors $x^4-15x^2y^2+9y^4$.

$$\begin{aligned}x^4-15x^2y^2+9y^4&=(x^4-6x^2y^2+9y^4)-9x^2y^2\\&=(x^2-3y^2)^2-(3xy)^2\\&=(x^2-3y^2+3xy)(x^2-3y^2-3xy).\end{aligned}$$

212. Expressions which can be put into the form $x^3\pm\frac{1}{y^3}$ may be separated into factors by the rules for resolving the sum or the difference of two cubes. [Art. 136.]

*Example* 1. $$\begin{aligned}\frac{8}{a^3}-27b^6&=\left(\frac{2}{a}\right)^3-(3b^2)^3\\&=\left(\frac{2}{a}-3b^2\right)\left(\frac{4}{a^2}+\frac{6b^2}{a}+9b^4\right).\end{aligned}$$

*Example* 2. Resolve $a^2x^3-\frac{8a^2}{y^3}-x^3+\frac{8}{y^3}$ into four factors.

$$\begin{aligned}a^2x^3-\frac{8a^2}{y^3}-x^3+\frac{8}{y^3}&=x^3(a^2-1)-\frac{8}{y^3}(a^2-1)\\&=(a^2-1)\left(x^3-\frac{8}{y^3}\right)\\&=(a+1)(a-1)\left(x-\frac{2}{y}\right)\left(x^2+\frac{2x}{y}+\frac{4}{y^2}\right)\end{aligned}$$

*Example* 3. Resolve $a^9-64a^3-a^6+64$ into six factors.

$$\begin{aligned}\text{The expression}&=a^3(a^6-64)-(a^6-64)\\&=(a^6-64)(a^3-1)\\&=(a^3+8)(a^3-8)(a^3-1)\\&=(a+2)(a^2-2a+4)(a-2)(a^2+2a+4)(a-1)(a^2+a+1).\end{aligned}$$

*Example* 4.

$$a(a-1)x^2-(a-b-1)xy-b(b+1)y^2=\{ax-(b+1)y\}\{(a-1)x+by\}.$$

NOTE. In examples of this kind the coefficients of $x$ and $y$ in the binomial factors can usually be guessed at once, and it only remains to verify the coefficient of the middle term.

213. From Example 2, Art. 52, we see that the quotient of

$$a^3+b^3+c^3-3abc \text{ by } a+b+c \text{ is } a^2+b^2+c^2-bc-ca-ab.$$

Thus $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$...(1).

This result is important and should be carefully remembered. We may note that the expression on the left consists of the sum of the cubes of three quantities $a$, $b$, $c$, diminished by 3 times the product $abc$. Whenever an expression admits of a similar arrangement, the above formula will enable us to resolve it into factors.

*Example* 1. Resolve into factors $a^3-b^3+c^3+3abc$.

$$\begin{aligned}a^3-b^3+c^3+3abc&=a^3+(-b)^3+c^3-3a(-b)c,\\&=(a-b+c)(a^2+b^2+c^2+bc-ca+ab),\end{aligned}$$

$-b$ taking the place of $b$ in formula (1).

*Example* 2.

$$\begin{aligned}x^3-8y^3-27-18xy&=x^3+(-2y)^3+(-3)^3-3x(-2y)(-3)\\&=(x-2y-3)(x^2+4y^2+9-6y+3x+2xy).\end{aligned}$$

## EXAMPLES XXVIII. a.

Resolve into factors:

1. $x^4+16x^2+256$.
2. $81a^4+9a^2b^2+b^4$.
3. $x^4+y^4-7x^2y^2$.
4. $m^4+n^4-18m^2n^2$.
5. $x^4-6x^2y^2+y^4$.
6. $4x^4+9y^4-93x^2y^2$.
7. $4m^4+9n^4-24m^2n^2$.
8. $9x^4+4y^4+11x^2y^2$.
9. $x^4-19x^2y^2+25y^4$.
10. $16a^4+b^4-28a^2b^2$.

**11.** $\frac{27}{a^3b^3}-1.$ **12.** $216a^3-\frac{b^3}{8}.$ **13.** $\frac{x^3}{125}+y^3.$

**14.** $\frac{m^3n^3}{729}-1.$ **15.** $\frac{a^3b^3}{125}+1000.$ **16.** $\frac{x^3}{512}-\frac{64}{x^3}.$

Resolve into two or more factors:

**17.** $x^2y+3xy^2-3x^3-y^3.$ **18.** $4mn^2-20n^3+45nm^2-9m^3.$
**19.** $ab(x^2+1)+x(a^2+b^2).$ **20.** $y^2z^2(x^4-1)+x^2(y^4-z^4).$
**21.** $a^3+(a+b)ax+bx^2.$ **22.** $pn(m^2+1)-m(p^2+n^2).$
**23.** $6bx(a^2+1)-a(4x^2+9b^2).$ **24.** $(2a^2+3y^2)x+(2x^2+3a^2)y.$
**25.** $(2x^2-3a^2)y+(2a^2-3y^2)x.$
**26.** $a(a-1)x^2+(2a^2-1)x+a(a+1).$
**27.** $3x^2-(4a+2b)x+a^2+2ab.$
**28.** $2a^2x^2-2(3b-4c)(b-c)y^2+abxy.$
**29.** $(a^2-3a+2)x^2+(2a^2-4a+1)x+a(a-1).$
**30.** $a(a+1)x^2+(a+b)xy-b(b-1)y^2.$
**31.** $b^3+c^3-1+3bc.$ **32.** $a^3+8c^3+1-6ac.$
**33.** $a^3+b^3+8c^3-6abc.$ **34.** $a^3-27b^3+c^3+9abc.$
**35.** $a^3-b^3-c^3-3abc.$ **36.** $8a^3+27b^3+c^3-18abc.$
**37.** Resolve $x^8+81x^4+6561$ into three factors.
**38.** Resolve $(a^4-2a^2b^2-b^4)^2-4a^4b^4$ into four factors.
**39.** Resolve $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$ into four factors.
**40.** Resolve $x^8-\frac{1}{256}$ into four factors.
**41.** Resolve $x^{16}-y^{16}$ into five factors.
**42.** Resolve $x^{18}-y^{18}$ into six factors.

Resolve into four factors:

**43.** $\frac{a^3}{x^2}-8x-a^3+8x^3.$ **44.** $x^9+x^3y^6-8x^6y^3-8y^9.$

**45.** $x^9+x^6+64x^3+64.$ **46.** $4a-9b+\frac{4b^3}{a^2}-\frac{9a^3}{b^2}.$

**47.** $\frac{xy^3}{72}-\frac{x^3y^5}{32}-\frac{1}{9x^2}+\frac{y^2}{4}.$ **48.** $x^6-25x^2+6\tfrac{1}{4}-\frac{1}{4}x^4.$

Resolve into five factors:

**49.** $x^7+x^4-16x^3-16.$ **50.** $16x^7-81x^3-16x^4+81.$

214. The actual processes of multiplication and division can often be partially or wholly avoided by a skilful use of factors.

It should be observed that the formulæ which the student has seen exemplified in the preceding pages are just as useful in their converse as in their direct application. Thus the formula for resolving into factors the difference of two squares is equally useful as enabling us to write down at once the product of the sum and the difference of two quantities.

*Example* 1. Multiply $2a+3b-c$ by $2a-3b+c$.

These expressions may be arranged thus:

$$2a+(3b-c) \text{ and } 2a-(3b-c).$$

Hence the product $=\{2a+(3b-c)\}\,\{2a-(3b-c)\}$

$$=(2a)^2-(3b-c)^2 \qquad \text{[Art. 133.]}$$
$$=4a^2-(9b^2-6bc+c^2)$$
$$=4a^2-9b^2+6bc-c^2.$$

*Example* 2. Multiply $(a^2+a+1)x-a-1$ by $(a-1)x-a^2+a-1$.

The product $=\{(a^2+a+1)x-(a+1)\}\,\{(a-1)x-(a^2-a+1)\}$

$$=(a^3-1)x^2-\{(a^4+a^2+1)+(a^2-1)\}\,x+(a^3+1)$$
$$=(a^3-1)x^2-(a^4+2a^2)\,x+a^3+1$$
$$=(a^3-1)x^2-a^2(a^2+2)\,x+a^3+1.$$

NOTE. The product of $a^2+a+1$ and $a^2-a+1$ is $a^4+a^2+1$ and should be written down without actual multiplication.

*Example* 3. Multiply $(3+x-2x^2)^2-(3-x+2x^2)^2$ ............ (1),

by $(3+x+2x^2)^2-(3-x-2x^2)^2$ ............ (2).

The expression (1)

$$=(3+x-2x^2+3-x+2x^2)\,(3+x-2x^2-3+x-2x^2)$$
$$=6\,(2x-4x^2)$$
$$=12x\,(1-2x).$$

The expression (2)

$$=(3+x+2x^2+3-x-2x^2)\,(3+x+2x^2-3+x+2x^2)$$
$$=6\,(2x+4x^2)$$
$$=12x\,(1+2x).$$

Therefore the product $=12x\,(1-2x)\times 12x\,(1+2x)$

$$=144x^2\,(1-4x^2).$$

*Example* 4. Divide the product of $2x^2+x-6$, and $6x^2-5x+1$ by $3x^2+5x-2$.

Denoting the division by means of a fraction, the required quotient

$$=\frac{(2x^2+x-6)(6x^2-5x+1)}{3x^2+5x-2}$$

$$=\frac{(2x-3)(x+2)(3x-1)(2x-1)}{(3x-1)(x+2)}$$

$$=(2x-3)(2x-1).$$

*Example* 5. Shew that $(2x+3y-z)^3+(3x+7y+z)^3$ is divisible by $5(x+2y)$.

The given expression is of the form $A^3+B^3$, and therefore has a divisor of the form $A+B$.

Therefore $(2x+3y-z)^3+(3x+7y+z)^3$

is divisible by $(2x+3y-z)+(3x+7y+z)$,

that is, by $5x+10y$,

or by $5(x+2y)$.

*Example* 6. Find the quotient when $a^3+8-5b(25b^2-6a)$ is divided by $a-5b+2$.

The expression $=a^3+8-125b^3+30ab$

$$=a^3+(-5b)^3+(2)^3-3\,.\,a(-5b)(2)$$

$$=(a-5b+2)(a^2+25b^2+4+10b-2a+5ab).$$

[Art. 213.]

$\therefore$ the quotient is $a^2+25b^2+4+10b-2a+5ab$.

*Example* 7. If $x+y=a$, and $x-y=b$ shew that

$$4(x^4-6x^2y^2+y^4)=6a^2b^2-a^4-b^4.$$

$$x^4-6x^2y^2+y^4=(x^2-2x^2y^2+y^4)-4x^2y^2$$

$$=(x^2-y^2)^2-\frac{1}{4}(4xy)^2$$

$$=\{(x+y)(x-y)\}^2-\frac{1}{4}\{(x+y)^2-(x-y)^2\}^2$$

$$=(ab)^2-\frac{1}{4}(a^2-b^2)^2;$$

$$\therefore\ 4(x^4-6x^2y^2+y^4)=4a^2b^2-(a^2-b^2)^2$$

$$=6a^2b^2-a^4-b^4.$$

## EXAMPLES XXVIII. b.

Find the product of

1. $2x-7y+3z$ and $2x+7y-3z$.
2. $3x^2-4xy+7y^2$ and $3x^2+4xy+7y^2$.
3. $5x^2+5xy-9y^2$ and $5x^2-5xy-9y^2$.
4. $7x^2-8xy+3y^2$ and $7x^2+8xy-3y^2$.
5. $x^3+2x^2y+2xy^2+y^3$ and $x^3-2x^2y+2xy^2-y^3$.
6. $(x+y)^2+2(x+y)+4$ and $(x+y)^2-2(x+y)+4$.
7. $(1+x+2x^2)^2-(1-x-2x^2)^2$ and $(1+x-2x^2)^2-(1-x+2x^2)^2$.
8. $(a^2+3a-1)^2-(a^2-3a-1)^2$ and $(a^2+a+1)^2-(a^2-a+1)^2$.
9. $x^3-4x^2+8x-8$ and $x^3+4x^2+8x+8$.
10. $x^3-6ax^2+18a^2x-27a^3$ and $x^3+6ax^2+18a^2x+27a^3$.
11. $x-a-\dfrac{x^2}{a}-\dfrac{a^2}{x}$ and $x+a+\dfrac{x^2}{a}-\dfrac{a^2}{x}$.
12. $(2x^2+3x+1)^2-(2x^2-3x-1)^2$ and $(x^2+6x-2)^2-(x^2-6x+2)^2$.

Find the continued product of

13. $x^2+ax+a^2$, $x^2-ax+a^2$, $x^4-a^2x^2+a^4$.
14. $1-x+x^2$, $1+x+x^2$, $1-x^2+x^4$, $1-x^4+x^8$.
15. $(a-x)^3$, $(a+x)^3$, $(a^2+x^2)^3$.
16. $(1-x)^2$, $(1+x)^2$, $(1+x^2)^2$, $(1+x^4)^2$.
17. $x^2+4x+3$, $x^2+x-2$, $x^2-5x+6$.
18. $x^2+2x-3$, $x^2-5x+6$, $x^2+3x+2$.
19. $x+2$, $x^2+2x+4$, $x-2$, $x^2-2x+4$.
20. Multiply the square of $a+3b$ by $a^2-6ab+9b^2$.
21. Multiply $\frac{1}{2}(a-b)^2+\frac{1}{2}(b-c)^2+\frac{1}{2}(c-a)^2$ by $a+b+c$.
22. Divide $(4x+3y-2z)^2-(3x-2y+3z)^2$ by $x+5y-5z$.
23. Divide $x^8+16a^4x^4+256a^8$ by $x^2+2ax+4a^2$.
24. Divide $(3x+4y-2z)^2-(2x+3y-4z)^2$ by $x+y+2z$.
25. Divide the product of $x^2+7x+10$ and $x+3$ by $x^2+5x+6$.

**26.** Divide $2x(x^2-1)(x+2)$ by $x^2+x-2$.

**27.** Divide $5x(x-11)(x^2-x-156)$ by $x^3+x^2-132x$.

**28.** Divide $x^6+19x^3-216$ by $(x^2-3x+9)(x-2)$.

**29.** Divide $(5x^2-3x-6)^2-(2x^2-7x+9)^2$ by the product of $3x-5$ and $x+3$.

**30.** Divide $a^9-b^9$ by the product of $a^2+ab+b^2$ and $a^6+a^3b^3+b^6$.

**31.** Divide $(x^3-3x^2y)^2-(3xy^2-y^3)^2$ by $(x-y)^3$.

**32.** Divide $(x^2-yz)^3+8y^3z^3$ by $x^2+yz$.

**33.** Divide $18xy+1+27x^3-8y^3$ by $1+3x-2y$.

**34.** Divide $(2x^2+3x-1)^2-(x^2+4x+5)^2$ by the product of $3x+4$ and $x+2$.

**35.** Divide the product of $6a^2-23a+20$ and $22a^2-81a+14$ by $33a^2-50a+8$.

**36.** Divide the product of $x^2+(a-b)x-ab$ and $x^2-(a-b)x-ab$ by $x^2+(a+b)x+ab$.

**37.** Divide $a^3-8y^3-9x(3x^2+2ay)$ by $a-3x-2y$.

**38.** Divide $27-8x^3-64y^3-72xy$ by $3-2(x+2y)$.

**39.** Shew that $(2x-3y+1)^3-(1-3x+2y)^3$ is divisible by $5(x-y)$.

**40.** Shew that the square of $x+1$ exactly divides

$$(x^3+x^2+4)^3-(x^3-2x+3)^3.$$

**41.** Shew that $2b+2d$ is a factor of the expression

$$(a+b+c+d)^3-(a-b+c-d)^3.$$

**42.** Shew that $(3x^2-7x+2)^3-(x^2-8x+8)^3$ is divisible by $2x-3$ and by $x+2$.

**43.** Shew that $(7x^2+3x-3)^3+(5x^2-4x-3)^3$ is divisible by $4x-3$ and by $3x+2$.

**44.** Shew that the sum of the cubes of $2x^2-5x-9$ and $x^2+6x-5$ is divisible by the product of $3x+7$ and $x-2$.

**45.** If $x+y=m$ and $x-y=n$, express $x^3+y^3$ in terms of $m$ and $n$.

**46.** If $x+y=m$ and $x-y=n$, shew that

$$16(x^4-7x^2y^2+y^4)=(5m^2-n^2)(5n^2-m^2).$$

**47.** Find the value of $x^4+x^2y^2+y^4$ when $x+y=2a$, $x-y=2b$.

**48.** If $x+y=2a$ and $x-y=2b$ prove that

$$x^4-23x^2y^2+y^4=(7a^2-3b^2)(7b^2-3a^2).$$

**49.** Find the value of $x^4-47x^2y^2+y^4$ in terms of $p$ and $q$ when $x+y=p$ and $x-y=q$.

**50.** Find the value of $x^4-2x^3y+2xy^3-y^4$ when $x=a+b$ and $y=a-b$.

# CHAPTER XXIX.

## MISCELLANEOUS THEOREMS AND EXAMPLES.

215. QUESTIONS upon the simple rules, e.g. Division, Highest Common Factor, Evolution &c., frequently occur which cannot be neatly and concisely worked without a ready use of factors and compound expressions. These we have hitherto excluded as unsuitable for the student until he has gained confidence and power by practice. We propose in the present chapter to bring together a miscellaneous collection of examples, for the most part not new in principle, but requiring some skill for their solution. The chapter will be found useful as a revision of the earlier chapters.

*Example.* Divide

$ax^4-(ap-b)x^3+(aq-bp-c)x^2+(bq+cp)x-cq$ by $ax^2+bx-c$.

$$\begin{array}{r|l|l}
ax^2+bx-c & ax^4-(ap-b)x^3+(aq-bp-c)x^2+(bq+cp)x-cq & x^2-px+q \\
 & \underline{ax^4+bx^3-cx^2\phantom{+(bq+cp)x-cq}} & \\
 & \phantom{ax^4}-apx^3+(aq-bp)x^2+(bq+cp)x & \\
 & \phantom{ax^4}\underline{-apx^3-bpx^2\phantom{+(aq)}+cpx} & \\
 & \phantom{ax^4-apx^3-b}aqx^2\phantom{(bq)}+bqx-cq & \\
 & \phantom{ax^4-apx^3-b}\underline{aqx^2\phantom{(bq)}+bqx-cq} &
\end{array}$$

NOTE. When the coefficients in divisor or dividend are compound quantities it is best to retain them in brackets throughout the work.

216. In the process of finding the highest common factor, by the rules explained in Chap. XVIII., every remainder that occurs in the course of the work contains the factor we are seeking. Hence when any one of the remainders admits of being resolved into factors, we may often shorten the work.

*Example* 1. Find the H.C.F. of $2x^3-(4a-3c)x^2+6(b-ac)x+9bc$ and $2x^3+(2a+3c)x^2+(3ac-4b)x-6bc$.

$$\begin{array}{r|l|l}
2x^3-(4a-3c)x^2+6(b-ac)x+9bc & 2x^3+(2a+3c)x^2+(3ac-4b)x-6bc & 1 \\
 & \underline{2x^3-(4a-3c)x^2+(6b-6ac)x+9bc} & \\
 & \phantom{2x^3-(4a}6ax^2+(9ac-10b)x-15bc &
\end{array}$$

$$\begin{aligned}
\text{Now the remainder} &= 6ax^2+9acx-10bx-15bc \\
&= 3ax(2x+3c)-5b(2x+3c) \\
&= (2x+3c)(3ax-5b).
\end{aligned}$$

Of these factors, $3ax-5b$ may clearly be rejected; therefore if there is a common factor it must be $2x+3c$. And by division, or by the method explained in Art. 152, we find that $2x+3c$ is a factor of each expression.

Hence the H. C. F. is $2x+3c$.

*Example* 2. Find the H. C. F. of $(a^2-2a)x^2+2(2a-1)x-a^2+1$ and $(a^2-a-2)x^2+(4a+1)x-a^2-a$.

Each of these expressions can be resolved into factors as explained in Art. 212, Ex. 4. Thus

$$\begin{aligned}(a^2-2a)x^2+2(2a-1)x-a^2+1&=a(a-2)x^2+2(2a-1)x-(a+1)(a-1)\\&=\{(a-2)x+(a+1)\}\{ax-(a-1)\}.\end{aligned}$$

$$\begin{aligned}(a^2-a-2)x^2+(4a+1)x-a^2-a&=(a-2)(a+1)x^2+(4a+1)x-a(a+1)\\&=\{(a-2)x+(a+1)\}\{(a+1)x-a\}.\end{aligned}$$

Hence the H. C. F. is $(a-2)x+a+1$.

## EXAMPLES XXIX. a.

Divide

1. $x^3+(a+b+c)x^2+(bc+ca+ab)x+abc$ by $x^2+(a+b)x+ab$.
2. $x^4-(5+a)x^3+(4+5a+b)x^2-(4a+5b)x+4b$ by $x^2-5x+4$.
3. $x^3-(a-b)x^2-(ab+2b^2)x+2ab^2$ by $x-b$.
4. $x^3-(p^2+3q^2)x+2p^2q-2q^3$ by $x+p+q$.
5. $x^3-(3mn+n^2)x+m(m^2-n^2)$ by $x+m+n$.
6. $a(a-1)x^2+(2a^2-1)x+a(a+1)$ by $(a-1)x+a$.
7. $x^4+(a+b)x^3+(a^2+ab+b^2)x^2+(a^3+b^3)x+a^2b^2$ by $x^2+ax+b^2$.
8. $2l^2x^2-2(3m-4n)(m-n)y^2+lmxy$ by $lx+2(m-n)y$.
9. $(a^2+a-2)x^2-(2a+1)xy-(a^2+a)y^2$ by $(a-1)x-ay$.
10. $x^3-(a-b-2)x^2-(ab+2a-2b)x-2ab$ by $(x-a)(x+2)$.
11. $(x+1)^8+4(x+1)^6+6(x+1)^4+4(x+1)^2+1$ by $x^2+2x+2$.
12. $(m+1)(bx+an)b^2x^2-(n+1)(mbx+a)a^2$ by $bx-a$.

Find the H.C.F. of

13. $(m^2-3m+2)x^2+(2m^2-4m+1)x+m(m-1)$ and
    $m(m-1)x^2+(2m^2-1)x+m(m+1)$.
14. $mpx^3+(mq-np)x^2-(mr+nq)x+nr$ and
    $max^3-(mc+na)x^2-(mb-nc)x+nb$.
15. $2ap^3+(3a-2b)p^2q+(a-3b)pq^2-bq^3$ and
    $3ap^3-(a+3b)p^2q+(2a+b)pq^2-2bq^3$.
16. $acx^3+(bc+ad)x^2+(bd+ac)x+bc$ and
    $2acx^3+(2bc-ad)x^2-(3ac+bd)x-3bc$.

17. $2a^2x^3-(4b+3)\,ax^2+2\,(3b-ac)\,x+3c$ and
$2a^2x^3+(2b-3)\,ax^2-(4ac+3b)\,x+6c.$

18. $2ax^3+(4a^2-1)\,bx^2-(2ab^2+3c)\,x-6abc$ and
$ax^3-(3-2a^2)\,bx^2+(2c-6ab^2)\,x+4abc.$

Find the L.C.M. of

19. $x^4-px^3+(q-1)\,x^2+px-q$ and $x^4-qx^3+(p-1)\,x^2+qx-p.$

20. $p\,(p+1)\,x^2+x-p\,(p-1)$ and $p\,(p+2)\,x^2+2x-p^2+1.$

21. $(a^2-5a+6)\,x^2+2\,(a-1)\,x-a\,(a+1)$ and
$a\,(a-3)\,x^2+12x-(a+1)\,(a+4).$

217. We add some miscellaneous questions in Evolution.

The *fourth* root of an expression is obtained by extracting the square root of the square root of the expression.

Similarly by successive applications of the rule for finding the square root, we may find the *eighth*, *sixteenth*...root. The *sixth* root of an expression is found by taking the cube root of the square root, or the square root of the cube root.

Similarly by combining the two processes for extraction of cube and square roots, certain other higher roots may be obtained.

*Example* 1. Find the fourth root of

$$81x^4-216x^3y+216x^2y^2-96xy^3+16y^4.$$

Extracting the square root by the rule we obtain $9x^2-12xy+4y^2$; and *by inspection*, the square root of this is $3x-2y$, which is the required fourth root.

*Example* 2. Find the sixth root of

$$\left(x^3-\frac{1}{x^3}\right)^2-6\left(x-\frac{1}{x}\right)\left(x^3-\frac{1}{x^3}\right)+9\left(x-\frac{1}{x}\right)^2.$$

By inspection, the square root of this is

$$\left(x^3-\frac{1}{x^3}\right)-3\left(x-\frac{1}{x}\right),$$

which may be written $\qquad x^3-3x+\frac{3}{x}-\frac{1}{x^3};$

and the cube root of this is $\qquad x-\frac{1}{x},$

which is the required sixth root.

218. In Chap. VI. we have given examples of inexact division. In a similar manner when an expression is not an exact square

or cube, we may perform the process of evolution, and obtain as many terms of the root as we please.

*Example.* To find four terms of the square root of $1+2x-2x^2$.

$$\begin{array}{l|l}
 & 1+2x-2x^2\left(1+x-\frac{3}{2}x^2+\frac{3}{2}x^3\right. \\
 & 1 \\ \hline
2+x & 2x-2x^2 \\
 & 2x+\;x^2 \\ \hline
2+2x-\frac{3}{2}x^2 & -3x^2 \\
 & -3x^2-3x^3+\frac{9}{4}x^4 \\ \hline
2+2x-3x^2+\frac{3}{2}x^3 & 3x^3-\frac{9}{4}x^4 \\
 & 3x^3+3x^4-\frac{9}{2}x^5+\frac{9}{4}x^6 \\ \hline
 & -\frac{21}{4}x^4+\frac{9}{2}x^5-\frac{9}{4}x^6.
\end{array}$$

Thus the required result is $1+x-\frac{3}{2}x^2+\frac{3}{2}x^3$.

*219. In Art. 124 we pointed out the similarity between the arithmetical and algebraical methods of extracting square and cube roots. We shall now shew that in extracting either the square or the cube root of any number, when a certain number of figures have been obtained by the common rule, that number may be nearly doubled by ordinary division.

*220. *If the square root of a number consists of* $2n+1$ *figures, when the first* $n+1$ *of these have been obtained by the ordinary method, the remaining* $n$ *may be obtained by division.*

Let $N$ denote the given number; $a$ the part of the square root already found, that is the first $n+1$ figures found by the common rule, with $n$ ciphers annexed; $x$ the remaining part of the root.

Then
$$\sqrt{N}=a+x;$$
$$\therefore\quad N=a^2+2ax+x^2;$$
$$\therefore\quad \frac{N-a^2}{2a}=x+\frac{x^2}{2a}\text{..............................}(1).$$

Now $N-a^2$ is the remainder after $n+1$ figures of the root, represented by $a$, have been found; and $2a$ is the divisor at the same stage of the work. We see from (1) that $N-a^2$ divided by $2a$ gives $x$, the rest of the quotient required, increased by $\frac{x^2}{2a}$. We shall shew that $\frac{x^2}{2a}$ is a *proper fraction*, so that by neglecting the remainder arising from the division, we obtain $x$, the rest of the root.

For $x$ contains $n$ figures, and therefore $x^2$ contains $2n$ figures at most; also $a$ is a number of $2n+1$ figures (the last $n$ of which are ciphers) and thus $2a$ contains $2n+1$ figures at least; and therefore $\frac{x^2}{2a}$ is a proper fraction.

From the above investigation, by putting $n=1$, we see that *two* at least of the figures of a square root must have been obtained in order that the method of division, which is employed to obtain the next figure of the square root, may give that figure correctly.

*Example.* Find the square root of 290 to five places of decimals.

$$\begin{array}{r|l}
 & \dot{2}9\dot{0}\,(17\cdot02 \\
 & 1 \\
27 & 190 \\
 & 189 \\
3402 & 10000 \\
 & \phantom{1}6804 \\
 & \phantom{1}3196
\end{array}$$

Here we have obtained four figures in the square root by the ordinary method. Three more may be obtained by division only, using $2\times1702$, that is 3404, for divisor, and 3196 as remainder. Thus

$$\begin{array}{l}
3404)\,31960\,(938 \\
\phantom{3404)\,}30636 \\
\phantom{3404)\,3}13240 \\
\phantom{3404)\,3}10212 \\
\phantom{3404)\,31}30280 \\
\phantom{3404)\,31}27232 \\
\phantom{3404)\,313}3048
\end{array}$$

And therefore to five places of decimals $\sqrt{290}=17\cdot02938$.

When the divisor consists of several digits, the method of contracted division may be employed with advantage.

Again, it may be noticed that in obtaining the second figure of the root, the division of 190 by 20 gives 9 for the next figure; this is too great, and the figure 7 has to be obtained tentatively. This is one of the modifications of the algebraical rule to which we referred in Art. 124.

*221. *If the cube root of a number consists of* 2n+2 *figures, when the first* n+2 *of these have been obtained by the ordinary method, the remaining* n *may be obtained by division.*

Let $N$ denote the given number; $a$ the part of the cube root already found, that is the first $n+2$ figures found by the common rule, with $n$ ciphers annexed; $x$ the remaining part of the root.

Then $\sqrt[3]{N}=a+x$;

$$\therefore \quad N=a^3+3a^2x+3ax^2+x^3;$$

$$\therefore \quad \frac{N-a^3}{3a^2}=x+\frac{x^2}{a}+\frac{x^3}{3a^2}\text{.............................}(1).$$

Now $N-a^3$ is the remainder after $n+2$ figures of the root, represented by $a$, have been found; and $3a^2$ is the divisor at the same stage of the work. We see from (1) that $N-a^3$ divided by $3a^2$ gives $x$, the rest of the quotient required, increased by $\frac{x^2}{a}+\frac{x^3}{3a^2}$. We shall shew that this expression is a *proper fraction,* so that by neglecting the remainder arising from the division, we obtain $x$, the rest of the root.

By supposition, $x$ is $<10^n$, and $a$ is $>10^{2n+1}$;

$$\therefore \quad \frac{x^2}{a} \text{ is } < \frac{10^{2n}}{10^{2n+1}}; \text{ that is, } < \frac{1}{10};$$

and $$\frac{x^3}{3a^2} \text{ is } < \frac{10^{3n}}{3\times 10^{4n+2}}; \text{ that is, } < \frac{1}{3\times 10^{n+1}};$$

hence $$\frac{x^2}{a}+\frac{x^3}{3a^2} \text{ is } < \frac{1}{10}+\frac{1}{3\times 10^{n+1}},$$

and is therefore a proper fraction.

## EXAMPLES XXIX. b.

Find the fourth roots of the following expressions:

1. $x^4-28x^3+294x^2-1372x+2401$.
2. $16-\frac{32}{m}+\frac{24}{m^2}-\frac{8}{m^3}+\frac{1}{m^4}$.
3. $a^4+8a^3x+16x^4+32ax^3+24a^2x^2$.
4. $1+4x+2x^2-8x^3-5x^4+8x^5+2x^6-4x^7+x^8$.
5. $1+8x+20x^2+8x^3-26x^4-8x^5+20x^6-8x^7+x^8$.

Find the sixth roots of the following expressions:

6. $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$.

7. $x^6-12ax^5+240a^4x^2-192a^5x+60a^2x^4-160a^3x^3+64a^6$.

8. $a^6-18a^5x+135a^4x^2-540a^3x^3+1215a^2x^4-1458ax^5+729x^6$.

Find the eighth roots of the following expressions:

9. $x^8-8x^7y+28x^6y^2-56x^5y^3+70x^4y^4$
$-56x^3y^5+28x^2y^6-8xy^7+y^8$.

10. $\{x^4+2(p-1)x^3+(p^2-2p-1)x^2-2(p-1)x+1\}^4$.

Find to four terms the square root of

11. $1+x$. 12. $1-2x$. 13. $4+2x$. 14. $1-x-x^2$.

15. $a^2-x$. 16. $x^2+a^2$. 17. $a^4-3x^2$. 18. $9a^2+12ax$.

Find to three terms the cube root of

19. $x^3-a^3$. 20. $8+x$. 21. $\dfrac{1}{a^3}+9x$.

22. $1-6x+21x^2$. 23. $27x^6-27x^5-18x^4$. 24. $64-48x+9x^2$.

## IDENTITIES AND TRANSFORMATIONS.

*222. DEFINITION. An **identity** is an algebraical statement which is true for all values of the letters involved in it.

*Examples.* $a^3+b^3=(a+b)(a^2-ab+b^2)$.

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-yz-zx-xy).$$

*223. An identity asserts that two expressions are always equal; and the proof of this equality is called "proving the identity." The method of procedure is to choose one of the expressions given, and to shew by successive transformations that it can be made to assume the form of the other.

*Example* 1. To prove that

$$bc(b-c)+ca(c-a)+ab(a-b)=-(b-c)(c-a)(a-b).$$

$$\begin{aligned}\text{The first side}&=bc(b-c)+c^2a-ca^2+a^2b-ab^2\\&=bc(b-c)+a^2(b-c)-a(b^2-c^2)\\&=(b-c)\{bc+a^2-a(b+c)\}\\&=(b-c)\{bc+a^2-ab-ac\}\\&=(b-c)\{a(a-b)-c(a-b)\}\\&=(b-c)(a-b)(a-c)\\&=-(b-c)(c-a)(a-b),\end{aligned}$$

changing the signs of the factor $a-c$, so as to preserve cyclic order. [Art. 172.]

The expression on the left-hand side can be readily put in the following forms:

$$a^2(b-c)+b^2(c-a)+c^2(a-b);$$
$$-\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\}.$$

Hence we have the following results:

$$bc(b-c)+ca(c-a)+ab(a-b)=-(b-c)(c-a)(a-b);$$
$$a^2(b-c)+b^2(c-a)+c^2(a-b)=-(b-c)(c-a)(a-b);$$
$$a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=(b-c)(c-a)(a-b).$$

These identities are of such frequent occurrence that they should be carefully noticed and remembered.

*Example* 2. If $2s=a+b+c$ prove that

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{s(s-a)(s-b)(s-c)}.$$

The first side $=\left(\frac{1}{s-a}+\frac{1}{s-b}\right)+\left(\frac{1}{s-c}-\frac{1}{s}\right)$

$$=\frac{s-b+s-a}{(s-a)(s-b)}+\frac{s-s+c}{s(s-c)}$$
$$=\frac{2s-a-b}{(s-a)(s-b)}+\frac{c}{s(s-c)}$$
$$=\frac{c}{(s-a)(s-b)}+\frac{c}{s(s-c)}$$
$$=c\left\{\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right\}$$
$$=\frac{c\{s^2-cs+s^2-as-bs+ab\}}{s(s-a)(s-b)(s-c)}$$
$$=\frac{c\{2s^2-s(a+b+c)+ab\}}{s(s-a)(s-b)(s-c)}$$
$$=\frac{abc}{s(s-a)(s-b)(s-c)},\text{ for } s(a+b+c)=s\,.\,2s=2s^2.$$

NOTE. Here $2s$ is a convenient abbreviation of $a+b+c$; and the reduction is much simplified by working in terms of $s$ instead of substituting its value at once. In examples of this kind, as a rule, the student should avoid substituting as long as the work can be carried on in terms of the symbol of abbreviation.

*Example* 3. If $x^2+u^2=2(xy+yz+zu-y^2-z^2)$
prove that $x=y=z=u$.

By transposing, we have

$$x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zu+u^2=0,$$

or $$(x-y)^2+(y-z)^2+(z-u)^2=0.$$

Now since the square of any quantity is always positive, each of the expressions $(x-y)^2$, $(y-z)^2$, $(z-u)^2$ is positive. Hence their sum cannot be zero unless each of them be separately equal to zero.

$$\therefore\ x-y=0,\ y-z=0,\ z-u=0;$$

or

$$x=y=z=u.$$

NOTE. The student should be careful to notice the difference between the conclusions to be drawn from the two statements

$$(x-a)^2+(y-b)^2=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

and

$$(x-a)(y-b)=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

From (1) we infer that *both* $x-a=0$ and $y-b=0$ *simultaneously*, while from (2) we infer that *either* $x-a=0$ *or* $y-b=0$.

## *EXAMPLES XXIX. c.

Prove the following identities:

1. $b(x^3+a^3)+ax(x^2-a^2)+a^3(x+a)=(a+b)(x+a)(x^2-ax+a^2)$.
2. $(ax+by)^2+(ay-bx)^2+c^2x^2+c^2y^2=(x^2+y^2)(a^2+b^2+c^2)$.
3. $(x+y)^3+3(x+y)^2z+3(x+y)z^2+z^3$
$=(x+z)^3+3(x+z)^2y+3(x+z)y^2+y^3$.
4. $(a+b+c)(ab+bc+ca)-abc=(a+b)(b+c)(c+a)$.
5. $(a+b+c)^2-a(b+c-a)-b(a+c-b)-c(a+b-c)$
$=2(a^2+b^2+c^2)$.
6. $(x-y)^3+(x+y)^3+3(x-y)^2(x+y)+3(x+y)^2(x-y)=8x^3$.
7. $x^2(y-z)+y^2(z-x)+z^2(x-y)+(y-z)(z-x)(x-y)=0$.
8. $a^3(b-c)+b^3(c-a)+c^3(a-b)=-(b-c)(c-a)(a-b)(a+b+c)$.
9. If $x+y+z=0$, prove that $x^3+y^3+z^3=3xyz$.
10. Prove that $(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b)$.

If $2s=a+b+c$, shew that

11. $(s-a)^2+(s-b)^2+(s-c)^2+s^2=a^2+b^2+c^2$.
12. $(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3$.
13. $16s(s-a)(s-b)(s-c)=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$.
14. $2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)$
$+c(s-a)(s-b)=abc$.

If $a+b+c=0$, shew that

15. $(2a-b)^3+(2b-c)^3+(2c-a)^3=3(2a-b)(2b-c)(2c-a)$.
16. $\dfrac{a^2}{2a^2+bc}+\dfrac{b^2}{2b^2+ca}+\dfrac{c^2}{2c^2+ab}=1$.
17. Prove that
$(x+y+z)^3+(x+y-z)^3+(x-y+z)^3+(x-y-z)^3=4x(x^2+3y^2+3z^2)$.

**18.** If $a+b+c=s$, prove that

$$(s-3a)^3+(s-3b)^3+(s-3c)^3-3(s-3a)(s-3b)(s-3c)=0.$$

**19.** If $X=b+c-2a$, $Y=c+a-2b$, $Z=a+b-2c$, find the value of $X^3+Y^3+Z^3-3XYZ$.

**20.** Find the value of $a(a^2+bc)+b(b^2+ac)-c(c^2-ab)$ when $a=\cdot 7$, $b=\cdot 08$, $c=\cdot 78$.

**21.** Prove that $(a-b)^2+(b-c)^2+(c-a)^2$

$$=2(c-b)(c-a)+2(b-a)(b-c)+2(a-b)(a-c).$$

**22.** Prove that $a^2(b^3-c^3)+b^2(c^3-a^3)+c^2(a^3-b^3)$

$$=(a-b)(b-c)(c-a)(ab+bc+ca)$$
$$=a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$$
$$=-[a^2b^2(a-b)+b^2c^2(b-c)+c^2a^2(c-a)].$$

**23.** If $(a+b)^2+(b+c)^2+(c+d)^2=4(ab+bc+cd)$ prove that $a=b=c=d$.

**24.** If $x=a+d$, $y=b+d$, $z=c+d$, prove that

$$x^2+y^2+z^2-yz-zx-xy=a^2+b^2+c^2-bc-ca-ab.$$

**25.** If $a+b+c=0$, prove that

$$\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}+\frac{1}{a^2+b^2-c^2}=0.$$

**26.** If $a+b+c=0$, simplify

$$\frac{b+c}{bc}(b^2+c^2-a^2)+\frac{c+a}{ca}(c^2+a^2-b^2)+\frac{a+b}{ab}(a^2+b^2-c^2).$$

**27.** Prove that the equation

$$(x-a)^2+(y-b)^2+(a^2+b^2-1)(x^2+y^2-1)=0,$$

is equivalent to the equation

$$(ax+by-1)^2+(bx-ay)^2=0\,;$$

hence shew that the only possible values of $x$ and $y$ are

$$\frac{a}{a^2+b^2},\ \frac{b}{a^2+b^2}.$$

**28.** If $2(x^2+a^2-ax)(y^2+b^2-by)=x^2y^2+a^2b^2$, shew that

$$(x-a)^2(y-b)^2+(bx-ay)^2=0,$$

and therefore that $x=a$, $y=b$ are the only possible solutions.

*224. We shall now give some further examples of fractions to illustrate the advantage of arranging expressions with regard to cyclic order. [Art. 172.]

*Example.* Find the value of

$$\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-c)(b-a)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}.$$

Changing the sign of one factor in each denominator, so as to preserve cyclic order, we get for the lowest common denominator,

$$(a-b)(b-c)(c-a)(x-a)(x-b)(x-c).$$

The whole expression has for its numerator

$$-[a(b-c)(x-b)(x-c)+\ldots\ldots+\ldots\ldots]$$

or $$-[a(b-c)\{x^2-(b+c)x+bc\}+\ldots\ldots+\ldots\ldots].$$

Arrange it according to powers of $x$; thus

$$\begin{aligned}\text{coefficient of } x^2 &= -\{a(b-c)+b(c-a)+c(a-b)\}\\ &=0;\\ \text{coefficient of } x &= \{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\}\\ &=(b-c)(c-a)(a-b); \qquad \text{[Art. 223.]}\end{aligned}$$

terms which do not contain $x$

$$\begin{aligned}&=-\{abc(b-c)+abc(c-a)+abc(a-b)\}\\ &=-abc\{b-c+c-a+a-b\}\\ &=0.\end{aligned}$$

Hence the expression $$=\frac{(b-c)(c-a)(a-b)x}{(b-c)(c-a)(a-b)(x-a)(x-b)(x-c)}$$

$$=\frac{x}{(x-a)(x-b)(x-c)}.$$

NOTE. In examples of this kind the work will be much facilitated if the student accustoms himself to readily writing down the following equivalents:

$$(b-c)+(c-a)+(a-b)=0.$$

$$a(b-c)+b(c-a)+c(a-b)=0.$$

$$a^2(b-c)+b^2(c-a)+c^2(a-b)=-(a-b)(b-c)(c-a).$$

$$bc(b-c)+ca(c-a)+ab(a-b)=-(a-b)(b-c)(c-a).$$

$$a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=(a-b)(b-c)(c-a).$$

Some of the identities in XXIX. c. may also be remembered with advantage.

## *EXAMPLES XXIX. d.

1. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}.$

2. $\dfrac{bc}{(a-b)(a-c)}+\dfrac{ca}{(b-c)(b-a)}+\dfrac{ab}{(c-a)(c-b)}.$

3. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}$

4. $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}$.

5. $\dfrac{a(b+c)}{(a-b)(c-a)}+\dfrac{b(a+c)}{(a-b)(b-c)}+\dfrac{c(a+b)}{(c-a)(b-c)}$.

6. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}$.

7. $\dfrac{bc}{a(a^2-b^2)(a^2-c^2)}+\dfrac{ca}{b(b^2-c^2)(b^2-a^2)}+\dfrac{ab}{c(c^2-a^2)(c^2-b^2)}$.

8. $\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{(x-c)(x-a)}{(b-c)(b-a)}+\dfrac{(x-a)(x-b)}{(c-a)(c-b)}$.

9. $\dfrac{bc(a+d)}{(a-b)(a-c)}+\dfrac{ca(b+d)}{(b-c)(b-a)}+\dfrac{ab(c+d)}{(c-a)(c-b)}$.

10. $\dfrac{1}{(a-b)(a-c)(x-a)}+\dfrac{1}{(b-c)(b-a)(x-b)}+\dfrac{1}{(c-a)(c-b)(x-c)}$.

11. $\dfrac{a^2}{(a-b)(a-c)(x+a)}+\dfrac{b^2}{(b-c)(b-a)(x+b)}+\dfrac{c^2}{(c-a)(c-b)(x+c)}$.

12. $a^2\dfrac{(a+b)(a+c)}{(a-b)(a-c)}+b^2\dfrac{(b+c)(b+a)}{(b-c)(b-a)}+c^2\dfrac{(c+a)(c+b)}{(c-a)(c-b)}$.

13. $\dfrac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{(b-c)^3+(c-a)^3+(a-b)^3}$.

14. $\dfrac{a^2(b-c)+b^2(c-a)+c^2(a-b)+2(a-b)(b-c)(c-a)}{(b-c)^3+(c-a)^3+(a-b)^3}$.

15. $\dfrac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$.

16. $\dfrac{a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3}{(a-b)(b-c)(c-a)}$.

17. $\dfrac{\dfrac{1}{a}(b-c)+\dfrac{1}{b}(c-a)+\dfrac{1}{c}(a-b)}{\dfrac{1}{a}\left(\dfrac{1}{b^2}-\dfrac{1}{c^2}\right)+\dfrac{1}{b}\left(\dfrac{1}{c^2}-\dfrac{1}{a^2}\right)+\dfrac{1}{c}\left(\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)}$.

18. $\dfrac{a^2\left(\dfrac{1}{c^2}-\dfrac{1}{b^2}\right)+b^2\left(\dfrac{1}{a^2}-\dfrac{1}{c^2}\right)+c^2\left(\dfrac{1}{b^2}-\dfrac{1}{a^2}\right)}{\dfrac{1}{bc}\left(\dfrac{1}{c}-\dfrac{1}{b}\right)+\dfrac{1}{ca}\left(\dfrac{1}{a}-\dfrac{1}{c}\right)+\dfrac{1}{ab}\left(\dfrac{1}{b}-\dfrac{1}{a}\right)}$.

***225.** *To find when* $x^3+px^2+qx+r$ ......................(1), *is divisible by* $x^2+ax+b$ ..........................(2).

Divide (1) by (2) in the ordinary way; thus

$$\begin{array}{r|l|l}
x^2+ax+b & x^3+px^2+qx+r & x+(p-a) \\
 & \underline{x^3+ax^2+bx} & \\
 & (p-a)x^2+\ (q-b)x+r & \\
 & \underline{(p-a)x^2+a(p-a)x+b(p-a)} & \\
 & \{(q-b)-a(p-a)\}x+r-b(p-a) \ldots\ldots\ldots\ldots(3). &
\end{array}$$

Now if the remainder be zero the division is exact. This is the case when

$$\{(q-b)-a(p-a)\}x+r-b(p-a)=0,$$

or

$$x=\frac{b(p-a)-r}{q-b-a(p-a)}.$$

Hence when $x$ has this value, (1) is divisible by (2).

But if in (3), $$q-b-a(p-a)=0,$$

and also $$r-b(p-a)=0,$$

the remainder is equal to zero *whatever value* x *may have.* Thus $x^3+px^2+qx+b$ is divisible by $x^2+ax+b$ for *all* values of $x$, provided that

$$q-b-a(p-a)=0,$$

and $$r-b(p-a)=0.$$

***226.** *To find the condition that* $x^2+px+q$ *may be a perfect square.*

It must be evident that any such general expression cannot be a perfect square unless some particular relation subsists between the coefficients $p$ and $q$. To find the necessary connection between $p$ and $q$ is the object of the present question.

Using the ordinary rule for square root, we have

$$\begin{array}{r|l}
 & x^2+px+q\ \Big(x+\frac{p}{2} \\
 & \underline{x^2} \\
2x+\frac{p}{2} & px+q \\
 & \underline{px+\frac{p^2}{4}} \\
 & q-\frac{p^2}{4}.
\end{array}$$

If therefore $x^2+px+q$ be a perfect square, the remainder, $q-\frac{p^2}{4}$, must be zero.

Hence $q-\frac{p^2}{4}=0$, or $p^2=4q$, is the condition required.

*227. *To prove that* $x^4+px^3+qx^2+rx+s$ *is a perfect square if* $\left(q-\frac{p^2}{4}\right)^2=4s$ *and* $r^2=p^2s$.

The square root must clearly be a trinomial expression of the form $x^2+lx+m$; if therefore we put

$$x^4+px^3+qx^2+rx+s=(x^2+lx+m)^2,$$

we have, on expanding the right-hand side,

$$x^4+px^3+qx^2+rx+s=x^4+2lx^3+x^2(l^2+2m)+2lmx+m^2.$$

Since this is to be true for all values of $x$, *we may assume that the coefficients of the like powers of* x *are the same;* hence

$$2l=p, \qquad l^2+2m=q,$$
$$2lm=r, \qquad m^2=s.$$

From these equations, by eliminating the unknown quantities $l$ and $m$, we shall obtain the necessary relations between $p$, $q$, $r$, and $s$.

Thus we have $$q-\frac{p^2}{4}=2m=2\sqrt{s},$$

and $$r=2lm=p\sqrt{s};$$

$$\therefore \left(q-\frac{p^2}{4}\right)^2=4s \text{ and } r^2=p^2s.$$

NOTE. The method of Art. 226 might have been used here. Also the method of the present article may be used to establish the results of Arts. 225 and 226.

*228. The proposition in the preceding article has been given to illustrate a useful method, which admits of very wide application. In the course of the proof we assume the truth of an important principle; namely,

*If two expressions involving* x *are identically equal, the coefficients of like powers in the two expressions are equal.*

The demonstration of this principle belongs to a more advanced part of the subject, and could not be discussed completely here.

*229. We shall now give general proofs of the statements made in Art. 55. We suppose $n$ to be positive and integral.

I. *To prove that* $x^n - y^n$ *is always divisible by* $x - y$.

Divide $x^n - y^n$ by $x - y$ till a remainder is obtained which does not involve $x$.

Let $Q$ be the quotient, and $R$ the remainder; then

$$x^n - y^n = Q(x-y) + R.$$

Since $R$ does not contain $x$, it will remain unaltered whatever value we give to $x$.

Put $x = y$, then $\qquad y^n - y^n = Q \times 0 + R$;

$\therefore R = 0$; that is, since there is no remainder, $x^n - y^n$ is divisible by $x - y$.

II. *To prove that* $x^n + y^n$ *is divisible by* $x + y$ *when* n *is odd, but not when* n *is even.*

With the same notation as before,

$$x^n + y^n = Q(x+y) + R.$$

Now since $R$ does not involve $x$, it will remain unaltered whatever value we give to $x$.

Put $x = -y$, then

$$(-y)^n + y^n = Q \times 0 + R;$$

that is, $$R = (-y)^n + y^n.$$

(1) if $n$ is odd, $(-y)^n + y^n = -y^n + y^n = 0$;

(2) if $n$ is even, $(-y)^n + y^n = \quad y^n + y^n = 2y^n$;

hence there is a remainder when $n$ is even, but none when $n$ is odd; which proves the proposition.

In like manner it may be proved that $x^n - y^n$ is divisible by $x + y$ when $n$ is even; and $x^n + y^n$ is *never* divisible by $x - y$.

By going through a few steps of the division, the form of the quotient in each case is easily determined. The results of the present article may be conveniently stated as follows:

(i) For all values of $n$,

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots\ldots + y^{n-1}).$$

(ii) When $n$ is odd,

$$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots\ldots + y^{n-1}).$$

(iii) When $n$ is even,

$$x^n - y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots\ldots - y^{n-1}).$$

*230. *If any algebraical expression*

$$x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\ldots\ldots+p_{n-1}x+p_n$$

*be divided by* $x-a$, *the remainder will be*

$$a^n+p_1a^{n-1}+p_2a^{n-2}+p_3a^{n-3}+\ldots\ldots+p_{n-1}a+p_n.$$

Divide the given expression by $x-a$ till a remainder is obtained which does not involve $x$. Let $Q$ be the quotient, and $R$ the remainder; then

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=Q(x-a)+R.$$

Since $R$ does not contain $x$, it will remain unaltered whatever value we give to $x$.

Put $x=a$, then

$$a^n+p_1a^{n-1}+p_2a^{n-2}+\ldots\ldots+p_{n-1}x+p_n=Q\times 0+R,$$

$$\therefore\quad R=a^n+p_1a^{n-1}+p_2a^{n-2}+\ldots\ldots+p_{n-1}a+p_n;$$

which proves the proposition.

From this it appears that when an algebraical expression is divided by $x-a$, the remainder can be obtained at once by writing $a$ in the place of $x$ in the given expression.

Again, the remainder is zero when the given expression is exactly divisible by $x-a$; hence we deduce the important proposition:

*If any algebraical expression involving* $x$ *become equal to* 0 *when* $a$ *is written for* $x$, *it will contain* $x-a$ *as a factor.*

*Example* 1. The remainder when $x^4-2x^3+x-7$ is divided by $x+2$ is

$$(-2)^4-2(-2)^3+(-2)-7;$$

that is, $\quad 16+16-2-7$, or 23.

Or the remainder may be found more shortly by substituting $x=-2$ in $[\{(x-2)x\}x+1]x-7$.

*Example* 2. Resolve into factors $x^3+3x^2-13x-15$.

By trial we find that this expression vanishes when $x=3$; hence $x-3$ is a factor.

$$\begin{aligned}\therefore\ x^3+3x^2-13x-15&=x^2(x-3)+6x(x-3)+5(x-3)\\&=(x-3)(x^2+6x+5)\\&=(x-3)(x+1)(x+5).\end{aligned}$$

Note. The only numerical values that need be substituted for $x$ are the factors of the last term of the expression. Thus, in the present case, by making trial of $-5$, we should have detected the factor $x+5$.

## *EXAMPLES XXIX. e.

Find the values of $x$ which will make each of the following expressions a perfect square:

**1.** $x^4+6x^3+13x^2+13x-1$. **2.** $x^4+6x^3+11x^2+3x+31$.

**3.** $x^4-2ax^3+(a^2+2b)\,x^2-3abx+2b^2$.

**4.** $4p^2x^4-4pqx^3+(q^2+2p^2)\,x^2-5pqx+\dfrac{p^2}{2}$.

**5.** $\dfrac{a^2x^6}{9}-\dfrac{abx^4}{2}+\dfrac{2acx^3}{3}+\dfrac{9b^2x^2}{16}-\dfrac{5bcx}{2}+6c^2$.

**6.** $x^4+2ax^3+3a^2x^2+cx+d$.

**7.** Find the conditions that $x^4-ax^3+bx^2-cx+1$ may be a perfect square for all values of $x$.

Find the values of $x$ which will make each of the following expressions a perfect cube:

**8.** $8x^3-36x^2+56x-39$. **9.** $\dfrac{x^6}{27}-\dfrac{a^2x^4}{3}+4a^4x^2-28a^6$.

**10.** $m^3x^6-9m^2nx^4+39mn^2x^2-51n^3$.

**11.** Find the relation between $b$ and $c$ in order that

$$x^3+3ax^2+bx+c$$

may be a perfect cube for all values of $x$.

**12.** Find the conditions that

$$x^6+3ax^5+3bx^4+a\,(6b-5a^2)\,x^3+3b\,(b-a^2)\,x^2+3cx+d$$

may be a perfect cube for all values of $x$.

**13.** What number must be added to $x^3+2x^2$ in order that the expression may be divisible by $x+4$.

**14.** If $x+a$ be a common factor of $x^2+px+q$ and $x^2+lx+m$, shew that $a=\dfrac{m-q}{l-p}$.

Resolve into factors:

**15.** $x^3-6x^2+11x-6$. **16.** $x^3-5x^2-2x+24$.

**17.** $x^3+9x^2+26x+24$. **18.** $x^3-x^2-41x+105$.

**19.** $x^3-39x+70$. **20.** $x^3-8x^2-31x-22$.

**21.** $6x^3+7x^2-x-2$. **22.** $6x^3+x^2-19x+6$.

Write down the quotient in the following cases:

**23.** $\dfrac{x^7+y^7}{x+y}$. **24.** $\dfrac{x^8-y^8}{x+y}$. **25.** $\dfrac{x^6-y^6}{x-y}$. **26.** $\dfrac{x^9-y^9}{x-y}$.

Find the square root of

**27.** $x^4+(2a-4)x^3+(a^2-2a+4)x^2+(2a^2-4a)x+a^2.$

**28.** $(a+1)^2x^4+(2a^2+2a)x^3+(3a^2-4a-6)x^2$
$+(2a^2-6a)x+a^2-6a+9.$

**29.** Find what values of $m$ make $3mx^2+(6m-12)x+8$ a perfect square.

**30.** If $4x^4+12x^3y+Px^2y^2+6xy^3+y^4$ is a perfect square, find $P$.

Without actual division shew that

**31.** $32x^{10}-33x^5+1$ is divisible by $x-1$.

**32.** $3x^4+5x^3-13x^2-20x+4$ ......................$x^2-4$.

**33.** $x^4+4x^3-5x^2-36x-36$ .................$x^2-x-6$.

Without actual division find the remainder when

**34.** $x^5-5x^2+5$ is divided by $x-5$.

**35.** $x^3-7x^2a+8xa^2+15a^3$ ............... $x+2a$.

**36.** If $ax^2-bx+c$ and $dx^3-bx+c$ have a common factor, then

$$a^3-abd+cd^2=0.$$

**37.** If $n$ be any positive integer, prove that $5^{2n}-1$ is always divisible by 24.

**38.** Shew that $1-x-x^n+x^{n+1}$ is exactly divisible by

$$1-2x+x^2.$$

**39.** If $x^3+px+r$ and $3x^2+p$ have a common factor, prove that

$$\frac{p^3}{27}+\frac{r^2}{4}=0.$$

**40.** Shew that if $x^n+py^n+qz^n$ is exactly divisible by

$$x^2-(ay+bz)x+abyz,$$

then $$\frac{p}{a^n}+\frac{q}{b^n}+1=0.$$

# CHAPTER XXX.

## THE THEORY OF INDICES.

231. HITHERTO all the definitions and rules with regard to indices have been based upon the supposition that they were positive integers; for instance

(1) $a^{14}=a\,.\,a\,.\,a\ldots\ldots$ to fourteen factors.

(2) $a^{14}\times a^{3}=a^{14+3}=a^{17}$.

(3) $a^{14}\div a^{3}=a^{14-3}=a^{11}$.

(4) $(a^{14})^{3}=a^{14\times 3}=a^{42}$.

The object of the present chapter is twofold: first, to give *general* proofs which shall establish the laws of combination in the case of all positive integral indices; secondly, to explain how, in strict accordance with these laws, intelligible meanings may be given to symbols whose indices are fractional, zero, or negative.

We shall begin by proving, directly from the definition of a positive integral index, three important propositions.

232. DEFINITION. When $m$ is a *positive integer*, $a^m$ stands for the product of $m$ factors each equal to $a$.

233. PROP. I. *To prove that* $a^m\times a^n=a^{m+n}$, *when* m *and* n *are positive integers.*

By definition, $a^m=a\,.\,a\,.\,a\ldots\ldots$ to $m$ factors;

$a^n=a\,.\,a\,.\,a\ldots\ldots$ to $n$ factors;

$$\therefore\ a^m\times a^n=(a\,.\,a\,.\,a\ldots\text{ to } m \text{ factors})\times(a\,.\,a\,.\,a\ldots\text{ to } n \text{ factors})$$
$$=a\,.\,a\,.\,a\ldots\text{ to } m+n \text{ factors}$$
$$=a^{m+n},\text{ by definition.}$$

COR. If $p$ is also a positive integer, then

$$a^m\times a^n\times a^p=a^{m+n+p};$$

and so for any number of factors.

234. **Prop. II.** *To prove that* $a^m \div a^n = a^{m-n}$, *when* m *and* n *are positive integers, and* $m > n$.

$$a^m \div a^n = \frac{a^m}{a^n} = \frac{a \,.\, a \,.\, a \ldots \text{ to } m \text{ factors}}{a \,.\, a \,.\, a \ldots \text{ to } n \text{ factors}}$$
$$= a \,.\, a \,.\, a \ldots \text{ to } m-n \text{ factors}$$
$$= a^{m-n}.$$

235. **Prop. III.** *To prove that* $(a^m)^n = a^{mn}$, *when* m *and* n *are positive integers.*

$$(a^m)^n = a^m \,.\, a^m \,.\, a^m \ldots \text{ to } n \text{ factors}$$
$$= (a \,.\, a \,.\, a \ldots \text{ to } m \text{ factors})(a \,.\, a \,.\, a \ldots \text{ to } m \text{ factors})\ldots\ldots$$

the bracket being repeated $n$ times,

$$= a \,.\, a \,.\, a \ldots \text{ to } mn \text{ factors}$$
$$= a^{mn}.$$

236. These are the fundamental laws of combination of indices, and they are proved directly from a definition which is intelligible only on the supposition that the indices are *positive* and *integral.*

But it is found convenient to use fractional and negative indices, such as $a^{\frac{4}{5}}$, $a^{-7}$; or, more generally, $a^{\frac{p}{q}}$, $a^{-n}$; and these have at present no intelligible meaning. For it is plain that the definition of $a^m$, [Art. 232] upon which we based the three propositions just proved, is no longer applicable when $m$ is *fractional*, or *negative.*

Now it is important that all indices, whether positive or negative, integral or fractional, should be governed by the same laws. We therefore determine meanings for symbols such as $a^{\frac{p}{q}}$, $a^{-n}$, in the following way: we assume that they conform to the fundamental law, $a^m \times a^n = a^{m+n}$, and accept the meaning to which this assumption leads us. It will be found that the symbols so interpreted will also obey the other laws enunciated in Props. II. and III.

237. *To find a meaning for* $a^{\frac{p}{q}}$, p *and* q *being positive integers.*

Since $a^m \times a^n = a^{m+n}$ is to be true for *all* values of $m$ and $n$, by replacing each of the indices $m$ and $n$ by $\frac{p}{q}$, we have

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}+\frac{p}{q}} = a^{\frac{2p}{q}}.$$

Similarly, $a^{\frac{p}{q}}\times a^{\frac{p}{q}}\times a^{\frac{p}{q}}=a^{\frac{2p}{q}}\times a^{\frac{p}{q}}=a^{\frac{2p}{q}+\frac{p}{q}}=a^{\frac{3p}{q}}$.

Proceeding in this way for 4, 5, ...... $q$ factors, we have

$$a^{\frac{p}{q}}\times a^{\frac{p}{q}}\times a^{\frac{p}{q}}\text{ ...... to } q \text{ factors}=a^{\frac{qp}{q}};$$

that is, $$(a^{\frac{p}{q}})^q=a^p.$$

Therefore, by taking the $q^{\text{th}}$ root,

$$a^{\frac{p}{q}}=\sqrt[q]{a^p},$$

or, in words, $a^{\frac{p}{q}}$ is equal to "the $q^{\text{th}}$ root of $a^p$."

*Examples.* (1) $x^{\frac{5}{7}}=\sqrt[7]{x^5}$.

(2) $a^{\frac{1}{3}}=\sqrt[3]{a}$.

(3) $4^{\frac{3}{2}}=\sqrt{4^3}=\sqrt{64}=8$.

(4) $a^{\frac{2}{3}}\times a^{\frac{5}{6}}=a^{\frac{2}{3}+\frac{5}{6}}=a^{\frac{3}{2}}$.

(5) $k^{\frac{a}{2}}\times k^{\frac{2}{3}}=k^{\frac{a}{2}+\frac{2}{3}}=k^{\frac{3a+4}{3}}$.

(6) $3a^{\frac{2}{3}}b^{\frac{1}{2}}\times 4a^{\frac{1}{6}}b^{\frac{5}{6}}=12a^{\frac{2}{3}+\frac{1}{6}}\,b^{\frac{1}{2}+\frac{5}{6}}=12a^{\frac{5}{6}}b^{\frac{4}{3}}$.

238. *To find a meaning for* a⁰.

Since $a^m\times a^n=a^{m+n}$ is to be true for *all* values of $m$ and $n$, by replacing the index $m$ by 0, we have

$$\begin{aligned} a^0\times a^n&=a^{0+n}\\ &=a^n,\\ \therefore\ a^0&=\frac{a^n}{a^n}\\ &=1. \end{aligned}$$

Hence *any quantity* with zero index is equivalent to 1.

[See Art. 47.]

*Example.* $x^{b-c}\times x^{c-b}=x^{b-c+c-b}=x^0=1$.

239. *To find a meaning for* $a^{-n}$.

Since $a^m\times a^n=a^{m+n}$ is to be true for *all* values of $m$ and $n$, by replacing the index $m$ by $-n$, we have

$$\begin{aligned} a^{-n}\times a^n&=a^{-n+n}\\ &=a^0\\ &=1. \end{aligned}$$

Hence $$a^{-n}=\frac{1}{a^n},$$

and $$a^n=\frac{1}{a^{-n}}.$$

Thus we see that any quantity may be removed from the numerator to the denominator, or vice-versâ, by merely changing the sign of the index.

*Examples.* (1) $x^{-3}=\dfrac{1}{x^3}$.

(2) $\dfrac{1}{y^{-\frac{1}{2}}}=y^{\frac{1}{2}}=\sqrt{y}$.

(3) $27^{-\frac{2}{3}}=\dfrac{1}{27^{\frac{2}{3}}}=\dfrac{1}{\sqrt[3]{(27)^2}}=\dfrac{1}{\sqrt[3]{3^6}}=\dfrac{1}{3^2}=\dfrac{1}{9}$.

240. *To prove that* $a^m \div a^n = a^{m-n}$ *for all values of* m *and* n.

$$\begin{aligned}a^m\div a^n&=a^m\times\frac{1}{a^n}\\&=a^m\times a^{-n}\\&=a^{m-n}, \text{ by the fundamental law.}\end{aligned}$$

*Examples.* (1) $a^3\div a^5=a^{3-5}=a^{-2}=\dfrac{1}{a^2}$.

(2) $c\div c^{-\frac{8}{5}}=c^{1+\frac{8}{5}}=c^{\frac{13}{5}}$.

(3) $x^{a-b}\div x^{a-c}=x^{a-b-(a-c)}=x^{c-b}$.

241. The method of finding a meaning for a symbol, as explained in the preceding articles, deserves careful attention. The usual algebraical process is to make choice of symbols, give them meanings, and then prove the rules for their combination. Here the process is reversed; the symbols are given, and the law to which they are to conform, and from this the meanings of the symbols are determined.

242. The following examples will illustrate the different principles we have established.

*Examples.* (1) $\dfrac{3a^{-2}}{5x^{-1}y}=\dfrac{3x}{5a^2y}$.

(2) $\dfrac{2a^{\frac{1}{2}}\times a^{\frac{2}{3}}\times 6a^{-\frac{7}{3}}}{9a^{-\frac{5}{3}}\times a^{\frac{3}{2}}}=\dfrac{4}{3}\,a^{\frac{1}{2}+\frac{2}{3}-\frac{7}{3}+\frac{5}{3}-\frac{3}{2}}=\dfrac{4}{3}\,a^{-1}=\dfrac{4}{3a}$.

(3) $\dfrac{\sqrt{x^3}\times\sqrt[3]{y^2}}{\sqrt[6]{y^{-2}}\times\sqrt[4]{x^6}}=\dfrac{x^{\frac{3}{2}}\times y^{\frac{2}{3}}}{y^{-\frac{1}{3}}\times x^{\frac{3}{2}}}=x^{\frac{3}{2}-\frac{3}{2}}y^{\frac{2}{3}+\frac{1}{3}}=x^0y=y.$

(4) $2\sqrt{a}+\dfrac{3}{a^{-\frac{1}{2}}}+a^{\frac{5}{2}}=2a^{\frac{1}{2}}+3a^{\frac{1}{2}}+a^{\frac{5}{2}}$

$$=5a^{\frac{1}{2}}+a^{\frac{5}{2}}$$

$$=a^{\frac{1}{2}}(5+a^2).$$

## EXAMPLES XXX. a.

Express with positive indices:

**1.** $2x^{-\frac{1}{4}}$. **2.** $3a^{-\frac{2}{3}}$. **3.** $4x^{-2}a^3$. **4.** $3\div a^{-2}$.

**5.** $\dfrac{1}{4a^{-2}}$. **6.** $\dfrac{1}{5x^{-\frac{1}{2}}}$. **7.** $\dfrac{3a^{-3}x^2}{5y^2c^{-4}}$. **8.** $\dfrac{x^ay^{-b}}{b^{-a}}$.

**9.** $2x^{\frac{1}{2}}\times 3x^{-1}$. **10.** $1\div 2a^{-\frac{1}{2}}$. **11.** $xy^2\times x^{-1}$. **12.** $a^{-2}x^{-1}\div 3x$.

**13.** $\dfrac{1}{\sqrt{x^3}}$. **14.** $\dfrac{1}{4\sqrt[5]{x^{-3}}}$. **15.** $\dfrac{2}{\sqrt{y^{-3}}}$. **16.** $\dfrac{\sqrt[4]{x^3}}{\sqrt{x^{-1}}}$.

**17.** $a^{-2}x^{-\frac{1}{2}}\div a^{-3}$. **18.** $\sqrt[3]{a^{-1}}\div\sqrt[3]{a}$. **19.** $\sqrt[5]{a^{-3}}\div\sqrt[5]{a^7}$.

Express with radical signs and positive indices:

**20.** $x^{\frac{3}{5}}$. **21.** $a^{-\frac{1}{2}}$. **22.** $5x^{-\frac{1}{2}}$. **23.** $2a^{-\frac{1}{x}}$.

**24.** $\dfrac{1}{2a^{\frac{1}{3}}}$. **25.** $\dfrac{2}{b^{-\frac{3}{4}}}$. **26.** $\dfrac{c^{-\frac{1}{3}}}{2}$. **27.** $\dfrac{1}{x^{-\frac{1}{x}}}$.

**28.** $a^{-\frac{1}{3}}\times 2a^{-\frac{1}{2}}$. **29.** $x^{-\frac{2}{3}}\div 2a^{-\frac{1}{2}}$. **30.** $7a^{-\frac{1}{2}}\times 3a^{-1}$.

**31.** $\dfrac{2a^{-2}}{a^{-\frac{3}{2}}}$. **32.** $\dfrac{a^{-\frac{1}{2}}}{3a}$. **33.** $\dfrac{4x^{-1}}{x^{-\frac{1}{3}}}$. **34.** $\dfrac{\sqrt[3]{x^{-a}}}{\sqrt[3]{x^2}}$.

**35.** $\sqrt[3]{a^2}\times\sqrt[2]{a^3}$. **36.** $\sqrt[5]{a^{-x}}\div\sqrt[5]{a^{-2x}}$. **37.** $\sqrt[2x]{x}\times\sqrt[x]{x^2}$.

**38.** $\sqrt[x]{x}\div\sqrt[2x]{x^3}$. **39.** $\sqrt[3x]{a^3}\div\sqrt[x]{a^2}$. **40.** $\sqrt[4]{a^n}\times\sqrt[3]{a^n}\div\sqrt[12]{a^{5n}}$.

Find the value of

**41.** $16^{\frac{3}{4}}$. **42.** $4^{-\frac{5}{2}}$. **43.** $125^{\frac{2}{3}}$. **44.** $8^{-\frac{2}{3}}$. **45.** $36^{-\frac{3}{2}}$.

**46.** $\dfrac{1}{25^{-2}}$. **47.** $243^{\frac{3}{5}}$. **48.** $\left(\dfrac{8}{27}\right)^{-\frac{1}{3}}$. **49.** $\left(\dfrac{81}{16}\right)^{\frac{3}{4}}$. **50.** $\left(\dfrac{32}{243}\right)^{-\frac{7}{5}}$.

**243.** *To prove that* $(a^m)^n = a^{mn}$ *is universally true for all values of* m *and* n.

Case I. Let $n$ be a *positive integer.*

Now, *whatever be the value of* m

$$\begin{aligned}(a^m)^n &= a^m \cdot a^m \cdot a^m \ldots \text{ to } n \text{ factors}\\ &= a^{m+m+m+\ldots \text{ to } n \text{ terms}}\\ &= a^{mn}.\end{aligned}$$

Case II. Let $m$ be unrestricted as before, and let $n$ be a *positive fraction.* Replacing $n$ by $\dfrac{p}{q}$, where $p$ and $q$ are *positive integers,* we have $(a^m)^n = (a^m)^{\frac{p}{q}}$.

Now the $q$th power of $(a^m)^{\frac{p}{q}} = \{(a^m)^{\frac{p}{q}}\}^q$

$$\begin{aligned}&= (a^m)^{\frac{p}{q}\cdot q}, && \text{[Case I.]}\\ &= (a^m)^p\\ &= a^{mp}. && \text{[Case I.]}\end{aligned}$$

Hence by taking the $q$th root of these equals,

$$\begin{aligned}(a^m)^{\frac{p}{q}} &= \sqrt[q]{a^{mp}}\\ &= a^{\frac{mp}{q}}. && \text{[Art. 237.]}\end{aligned}$$

Case III. Let $m$ be unrestricted as before, and let $n$ be *any negative quantity.* Replacing $n$ by $-r$, where $r$ is *positive,* we have

$$\begin{aligned}(a^m)^n = (a^m)^{-r} &= \frac{1}{(a^m)^r}, && \text{[Art. 239.]}\\ &= \frac{1}{a^{mr}}, && \text{[Case II.]}\\ &= a^{-mr} = a^{mn}.\end{aligned}$$

Hence Prop. III., Art. 235, $(a^m)^n = a^{mn}$ has been shewn to be universally true.

*Examples.* (1) $(b^{\frac{2}{3}})^{\frac{6}{7}} = b^{\frac{2}{3}\times\frac{6}{7}} = b^{\frac{4}{7}}$.

(2) $\{(x^{-2})^3\}^{-4} = (x^{-6})^{-4} = x^{24}$.

(3) $(x^{\frac{1}{a-c}})^{a^2-c^2} = x^{\frac{1}{a-c}\times(a^2-c^2)} = x^{a+c}$.

244. *To prove that* $(ab)^n = a^n b^n$, *whatever be the value of* $n$; $a$ *and* $b$ *being any quantities whatever.*

Case I. Let $n$ be a *positive integer.*

Now $(ab)^n = ab \,.\, ab \,.\, ab \ldots$ to $n$ factors

$= (a \,.\, a \,.\, a \ldots$ to $n$ factors$)\,(b \,.\, b \,.\, b \ldots$ to $n$ factors$)$

$= a^n b^n$.

Case II. Let $n$ be a *positive fraction.* Replacing $n$ by $\frac{p}{q}$, where $p$ and $q$ are *positive integers,* we have $(ab)^n = (ab)^{\frac{p}{q}}$.

Now the $q$th power of $(ab)^{\frac{p}{q}} = \{(ab)^{\frac{p}{q}}\}^q$

$= (ab)^p$, [Art. 243.]

$= a^p b^p$

$= (a^{\frac{p}{q}} b^{\frac{p}{q}})^q$. [Case I.]

Taking the $q$th root, $(ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}$.

Case III. Let $n$ have *any negative value.* Replacing $n$ by $-r$, where $r$ is positive,

$$(ab)^n = (ab)^{-r} = \frac{1}{(ab)^r}$$

$$= \frac{1}{a^r b^r} = a^{-r} b^{-r}$$

$$= a^n b^n.$$

Hence the proposition is proved universally.

The result we have just proved may be expressed in a verbal form by saying that the index of a product may be *distributed* over its *factors.*

Note. An index is not distributive over the *terms* of an expression. Thus $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$ is not equal to $a + b$. Again $(a^2 + b^2)^{\frac{1}{2}}$ is equal to $\sqrt{a^2 + b^2}$, and cannot be further simplified.

*Examples.* (1) $(yz)^{a-c}(zx)^{c}(xy)^{-c}=y^{a-c}z^{a-c}z^{c}x^{c}x^{-c}y^{-c},$

$$=y^{a-2c}z^{a}.$$

(2) $\{(a-b)^{k}\}^{-l}\times\{(a+b)^{-k}\}^{l}=(a-b)^{-kl}\times(a+b)^{-kl}$

$$=\{(a-b)(a+b)\}^{-kl},$$

$$=(a^2-b^2)^{-kl}.$$

**245.** It should be observed that in the proof of Art. 244 the quantities $a$ and $b$ are *wholly unrestricted*, and may themselves involve indices.

*Examples.* (1) $(x^{\frac{1}{2}}y^{-\frac{1}{2}})^{\frac{4}{3}}\div(x^2y^{-1})^{-\frac{1}{3}}=x^{\frac{2}{3}}y^{-\frac{2}{3}}\div x^{-\frac{2}{3}}y^{\frac{1}{3}}$

$$=x^{\frac{4}{3}}y^{-1}.$$

(2) $\left(\dfrac{a^{\frac{2}{3}}\sqrt{b^{-1}}}{b\sqrt[3]{a^{-2}}}\div\sqrt{\dfrac{a\sqrt{b^{-4}}}{b\sqrt{a^{-2}}}}\right)^6=\left(\dfrac{a^{\frac{2}{3}}b^{-\frac{1}{2}}}{ba^{-\frac{2}{3}}}\div\sqrt{\dfrac{ab^{-2}}{ba^{-1}}}\right)^6$

$$=(a^{\frac{4}{3}}b^{-\frac{3}{2}}\div\sqrt{a^2b^{-3}})^6$$

$$=(a^{\frac{4}{3}}b^{-\frac{3}{2}}\div ab^{-\frac{3}{2}})^6$$

$$=(a^{\frac{1}{3}})^6=a^2.$$

## EXAMPLES XXX. b.

Simplify and express with positive indices:

1. $(\sqrt{a^2b^3})^6$. 2. $(\sqrt[9]{x^{-4}y^3})^{-3}$. 3. $(x^ay^{-b})^3\times(x^3y^2)^{-a}$.

4. $\left(\dfrac{16x^2}{y^{-2}}\right)^{-\frac{1}{4}}$. 5. $\left(\dfrac{27x^3}{8a^{-3}}\right)^{-\frac{2}{3}}$. 6. $\left(\dfrac{a^{-\frac{1}{2}}}{4c^2}\right)^{-2}$.

7. $\left\{\sqrt[4]{\left(x^{-\frac{2}{3}}y^{\frac{1}{2}}\right)^3}\right\}^{-\frac{2}{3}}$. 8. $\sqrt[4]{x\sqrt[3]{x^{-1}}}$. 9. $(4a^{-2}\div 9x^2)^{-\frac{1}{2}}$.

10. $(x\div\sqrt[n]{x})^n$. 11. $\left(x\times\sqrt[n]{x^{-\frac{1}{n}}}\right)^{\frac{n^2}{1-n}}$. 12. $(\sqrt[b]{x^b}\div\sqrt[a]{x})^{\frac{1}{1-a}}$.

13. $\sqrt{a^{-2}b}\times\sqrt[3]{ab^{-3}}$. 14. $\sqrt[3]{ab^{-1}c^{-2}}\times(a^{-1}b^{-2}c^{-4})^{-\frac{1}{6}}$.

15. $\sqrt[6]{a^{4b}x^6}\times(a^{\frac{2}{3}}x^{-1})^{-b}$. 16. $\sqrt[3]{x^{-1}\sqrt{y^3}}\div\sqrt{y\sqrt[3]{x}}$.

17. $(a^{-\frac{1}{2}}\sqrt[3]{x})^{-3} \times \sqrt{x^{-2}\sqrt{a^{-6}}}$.

18. $\sqrt[n]{a^{n+k}b^{2n-k}} \div (a^{\frac{1}{n}}b^{-\frac{1}{n}})^k$.

19. $\sqrt[3]{(a+b)^5} \times (a+b)^{-\frac{2}{3}}$.

20. $\{(x-y)^{-3}\}^n \div \{(x+y)^n\}^3$.

21. $\left(\dfrac{a^{-2}b}{a^3b^{-4}}\right)^{-3} \div \left(\dfrac{ab^{-1}}{a^{-3}b^2}\right)^5$.

22. $\left\{\dfrac{\sqrt[3]{a}}{\sqrt[4]{b^{-1}}} \cdot \left(\dfrac{b^{\frac{1}{4}}}{a^{\frac{1}{3}}}\right)^2 \div \dfrac{a^{-\frac{1}{3}}}{b^{-\frac{1}{2}}}\right\}^6$.

23. $\left(a^{-\frac{1}{2}}x^{\frac{1}{3}}\sqrt{ax^{-\frac{1}{3}}\sqrt[4]{x^{\frac{4}{3}}}}\right)^{\frac{1}{3}}$.

24. $\sqrt[4]{(a+b)^6} \times (a^2-b^2)^{-\frac{1}{2}}$.

25. $\left(\dfrac{a^{-3}}{b^{-\frac{2}{3}}c}\right)^{-\frac{3}{2}} \div \left(\dfrac{\sqrt{a^{-\frac{1}{2}}.\sqrt[6]{b^3}}}{a^2c^{-1}}\right)^{-2}$.

26. $\left(\dfrac{a^{-\frac{2}{3}}x^{\frac{1}{2}}}{x^{-1}a}\right)^2 \div \sqrt[3]{\dfrac{a^{-1}}{x^{-3}}}$.

27. $\left(\sqrt[5]{\dfrac{a^{\frac{1}{2}}x^{-2}}{x^{\frac{1}{2}}a^{-2}}} \times \sqrt[3]{\dfrac{a\sqrt{x}}{x^{-1}\sqrt{a}}}\right)^{-4}$.

28. $\dfrac{\sqrt[3]{(a^3b^3+a^6)}}{\sqrt[3]{(b^6-a^3b^3)^{-1}}}$.

29. $(a^{n^2-1})^{\frac{n}{n+1}} + \dfrac{\sqrt[n]{a^{2n}}}{a}$.

30. $(x^{\frac{n}{n+1}})^{n^2-1} + \dfrac{\sqrt{x^{2n}}}{x}$.

31. $\left\{\dfrac{a^{p-q}}{\sqrt[q]{a^{q^2-pq}}} \times a^{2(p-q)}\right\}^n$.

32. $(x^{\frac{a}{b}}y^{-1})^b \div \left(\dfrac{x^{a^2-b^2}}{y^{ab+b^2}}\right)^{\frac{1}{a+b}}$.

33. $\left(\dfrac{x^{-2}y^3}{x^3y^{-2}}\right)^{-\frac{1}{5}} \times \left(\dfrac{y^3x^{-3}}{x^3y^{-3}}\right)^{-1}$.

34. $\left(\dfrac{y^{-3}}{x^{\frac{2}{7}}z^{-1}}\right)^{-\frac{3}{2}} \times \left(\dfrac{y^{\frac{14}{3}}x^{-1}}{z^{-\frac{21}{4}}}\right)^{\frac{2}{7}}$.

35. $\dfrac{2^n \times (2^{n-1})^n}{2^{n+1} \times 2^{n-1}} \times \dfrac{1}{4^{-n}}$.

36. $\dfrac{2^{n+1}}{(2^n)^{n-1}} \div \dfrac{4^{n+1}}{(2^{n-1})^{n+1}}$.

**246.** Since the index-laws are universally true, all the ordinary operations of multiplication, division, involution and evolution are applicable to expressions which contain fractional and negative indices.

**247.** In Art. 121, we pointed out that the descending powers of $x$ are

$$\ldots\ldots x^3,\ x^2,\ x,\ 1,\ \frac{1}{x},\ \frac{1}{x^2},\ \frac{1}{x^3},\ \ldots\ldots$$

A reason for this may be seen if we write these terms in the form

$$\ldots\ldots x^3,\ x^2,\ x^1,\ x^0,\ x^{-1},\ x^{-2},\ x^{-3},\ \ldots\ldots$$

*Example* 1. Multiply $3x^{-\frac{1}{3}}+x+2x^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-2$.

Arrange in descending powers of $x$.

$$\begin{array}{l}
x+2x^{\frac{2}{3}}+3x^{-\frac{1}{3}} \\
x^{\frac{1}{3}}-2 \\
\hline
x^{\frac{4}{3}}+2x+3 \\
\qquad -2x-4x^{\frac{2}{3}}-6x^{-\frac{1}{3}} \\
\hline
x^{\frac{4}{3}}-4x^{\frac{2}{3}}+3-6x^{-\frac{1}{3}}.
\end{array}$$

*Example* 2. Divide $16a^{-3}-6a^{-2}+5a^{-1}+6$ by $1+2a^{-1}$.

$$\begin{array}{r|l}
2a^{-1}+1 & 16a^{-3}-\ 6a^{-2}+\ 5a^{-1}+6\,(8a^{-2}-7a^{-1}+6 \\
 & 16a^{-3}+\ 8a^{-2} \\
\hline
 & \qquad -14a^{-2}+\ 5a^{-1} \\
 & \qquad -14a^{-2}-\ 7a^{-1} \\
\hline
 & \qquad\qquad\qquad 12a^{-1}+6 \\
 & \qquad\qquad\qquad 12a^{-1}+6. \\
\hline
\end{array}$$

*Example* 3. Find the square root of

$$\frac{4x^2}{y}+\frac{\sqrt{x^3}}{y^{-\frac{1}{2}}}-2x+\frac{y}{4}+x^3-4\sqrt{(x^5y^{-1})}.$$

Getting rid of the radical signs and negative indices, and then arranging in descending powers of $y$ we have

$$\begin{array}{l|l}
 & \frac{y}{4}+y^{\frac{1}{2}}x^{\frac{3}{2}}+x^3-2x-\frac{4x^{\frac{5}{2}}}{y^{\frac{1}{2}}}+\frac{4x^2}{y}\left(\frac{y^{\frac{1}{2}}}{2}+x^{\frac{3}{2}}-\frac{2x}{y^{\frac{1}{2}}}\right. \\
 & \frac{y}{4} \\
\hline
y^{\frac{1}{2}}+x^{\frac{3}{2}} & y^{\frac{1}{2}}x^{\frac{3}{2}}+x^3 \\
 & y^{\frac{1}{2}}x^{\frac{3}{2}}+x^3 \\
\hline
y^{\frac{1}{2}}+2x^{\frac{3}{2}}-\frac{2x}{y^{\frac{1}{2}}} & -2x-\frac{4x^{\frac{5}{2}}}{y^{\frac{1}{2}}}+\frac{4x^2}{y} \\
 & -2x-\frac{4x^{\frac{5}{2}}}{y^{\frac{1}{2}}}+\frac{4x^2}{y}. \\
\hline
\end{array}$$

## EXAMPLES XXX. c.

1. Multiply $3x^{\frac{1}{3}}-5+8x^{-\frac{1}{3}}$ by $4x^{\frac{1}{3}}+3x^{-\frac{1}{3}}$.

2. Multiply $3a^{\frac{3}{5}}-4a^{\frac{1}{5}}-a^{-\frac{1}{5}}$ by $3a^{\frac{1}{5}}+a^{-\frac{1}{5}}-6a^{-\frac{3}{5}}$.

3. Find the product of $c^x+2c^{-x}-7$ and $5-3c^{-x}+2c^x$.

4. Find the product of $5+2x^{2a}+3x^{-2a}$ and $4x^a-3x^{-a}$.

5. Divide $21x+x^{\frac{2}{3}}+x^{\frac{1}{3}}+1$ by $3x^{\frac{1}{3}}+1$.

6. Divide $15a-3a^{\frac{1}{3}}-2a^{-\frac{1}{3}}+8a^{-1}$ by $5a^{\frac{2}{3}}+4$.

7. Divide $16a^{-3}+6a^{-2}+5a^{-1}-6$ by $2a^{-1}-1$.

8. Divide $5b^{\frac{2}{3}}-6b^{\frac{1}{3}}-4b^{-\frac{2}{3}}-4b^{-\frac{1}{3}}-5$ by $b^{\frac{1}{6}}-2b^{-\frac{1}{6}}$.

9. Divide $21a^{3x}+20-27a^x-26a^{2x}$ by $3a^x-5$.

10. Divide $8c^{-n}-8c^n+5c^{3n}-3c^{-3n}$ by $5c^n-3c^{-n}$.

Find the square root of

11. $9x-12x^{\frac{1}{2}}+10-4x^{-\frac{1}{2}}+x^{-1}$.

12. $25a^{\frac{4}{3}}+16-30a-24a^{\frac{1}{3}}+49a^{\frac{2}{3}}$.

13. $4x^n+9x^{-n}+28-24x^{-\frac{n}{2}}-16x^{\frac{n}{2}}$.

14. $12a^x+4-6a^{3x}+a^{4x}+5a^{2x}$.

15. Multiply $a^{\frac{3}{2}}-8a^{-\frac{3}{2}}+4a^{-\frac{1}{2}}-2a^{\frac{1}{2}}$ by $4a^{-\frac{3}{2}}+a^{\frac{1}{2}}+4a^{-\frac{1}{2}}$.

16. Multiply $1-2\sqrt[3]{x}-2x^{\frac{1}{2}}$ by $1-\sqrt[6]{x}$.

17. Multiply $2\sqrt[3]{a^5}-a^{\frac{1}{3}}-\frac{3}{a}$ by $2a-3\sqrt[3]{\frac{1}{a}}-a^{-\frac{5}{3}}$.

18. Divide $\sqrt[3]{x^2}+2x^{\frac{1}{3}}-16x^{-\frac{2}{3}}-\frac{32}{x}$ by $x^{\frac{1}{6}}+4x^{-\frac{1}{6}}+\frac{4}{\sqrt{x}}$.

19. Divide $1-\sqrt{a}-\frac{2}{a^{-1}}+2a^2$ by $1-a^{\frac{1}{2}}$.

20. Divide $4\sqrt[3]{x^2}-8x^{\frac{1}{3}}-5+\frac{10}{\sqrt[3]{x}}+3x^{-\frac{2}{3}}$ by $2x^{\frac{5}{12}}-\sqrt[12]{x}-\frac{3}{\sqrt[4]{x}}$.

Find the square root of

21. $9x^{-4}-18x^{-3}\sqrt{y}+\dfrac{15y}{x^2}-6\sqrt{\left(\dfrac{y^3}{x^2}\right)}+y^2.$

22. $4\sqrt{x^3}-12\sqrt[4]{(x^3y)}+25\sqrt{y}-24\sqrt[4]{\left(\dfrac{y^3}{x^3}\right)}+16x^{-\frac{3}{2}}y.$

23. $81\left(\dfrac{\sqrt[3]{x^4}}{y^2}+1\right)+36\dfrac{x^{\frac{1}{3}}}{\sqrt{y}}\left(x^{\frac{2}{3}}y^{-1}-1\right)-158\dfrac{\sqrt[3]{x^2}}{y}.$

24. $\dfrac{x^{-2}}{16}+1+\dfrac{9}{\sqrt[3]{y^{-2}}}+\dfrac{1-3\sqrt[3]{y}}{2x}-6\sqrt[3]{y}.$

248. The following examples will illustrate the formulæ of earlier chapters when applied to expressions involving fractional and negative indices.

*Example* 1. $\left(a^{\frac{h}{k}}-b^{\frac{p}{q}}\right)\left(a^{-\frac{h}{k}}+b^{-\frac{p}{q}}\right)=a^{\frac{h}{k}-\frac{h}{k}}-a^{-\frac{h}{k}}b^{\frac{p}{q}}+a^{\frac{h}{k}}b^{-\frac{p}{q}}-b^{\frac{p}{q}-\frac{p}{q}}$

$$=1-a^{-\frac{h}{k}}b^{\frac{p}{q}}+a^{\frac{h}{k}}b^{-\frac{p}{q}}-1$$

$$=a^{\frac{h}{k}}b^{-\frac{p}{q}}-a^{-\frac{h}{k}}b^{\frac{p}{q}}.$$

*Example* 2. Multiply $2x^{2p}-x^p+3$ by $2x^{2p}+x^p-3$.

The product $=\{2x^{2p}-(x^p-3)\}\{2x^{2p}+(x^p-3)\}$

$$=(2x^{2p})^2-(x^p-3)^2$$

$$=4x^{4p}-x^{2p}+6x^p-9.$$

*Example* 3. The square of $3x^{\frac{1}{2}}-2-x^{-\frac{1}{2}}$

$$=9x+4+x^{-1}-2\,.\,3x^{\frac{1}{2}}\,.\,2-2\,.\,3x^{\frac{1}{2}}\,.\,x^{-\frac{1}{2}}+2\,.\,2\,.\,x^{-\frac{1}{2}}$$

$$=9x+4+x^{-1}-12x^{\frac{1}{2}}-6+4x^{-\frac{1}{2}}$$

$$=9x-12x^{\frac{1}{2}}-2+4x^{-\frac{1}{2}}+x^{-1},$$

by collecting like terms and rearranging.

*Example* 4. Divide $a^{\frac{3n}{2}}+a^{-\frac{3n}{2}}$ by $a^{\frac{n}{2}}+a^{-\frac{n}{2}}$.

The quotient $=\left(a^{\frac{3n}{2}}+a^{-\frac{3n}{2}}\right)\div\left(a^{\frac{n}{2}}+a^{-\frac{n}{2}}\right)$

$$=\left\{\left(a^{\frac{n}{2}}\right)^3+\left(a^{-\frac{n}{2}}\right)^3\right\}\div\left(a^{\frac{n}{2}}+a^{-\frac{n}{2}}\right)$$

$$=\left(a^{\frac{n}{2}}\right)^2-a^{\frac{n}{2}}\,.\,a^{-\frac{n}{2}}+\left(a^{-\frac{n}{2}}\right)^2$$

$$=a^n-1+a^{-n}.$$

## EXAMPLES XXX. d.

*Write down* the value of

**1.** $(x^{\frac{1}{2}}-7)(x^{\frac{1}{2}}+3)$. **2.** $(4x-5x^{-1})(4x+3x^{-1})$.

**3.** $(7x-9y^{-1})(7x+9y^{-1})$. **4.** $(x^m-y^n)(x^{-m}+y^{-n})$.

**5.** $(a^x-2a^{-x})^2$. **6.** $(a^x+a^{\frac{1}{x}})^2$. **7.** $\left(x^{\frac{a}{2}}-\frac{1}{2}x^{-a}\right)^2$.

**8.** $(5x^ay^b-3x^{-a}y^{-b})(4x^ay^b+5x^{-a}y^{-b})$.

**9.** $\left(\frac{1}{3}a^{\frac{1}{3}}-a^{-\frac{1}{3}}\right)^2$. **10.** $(3x^ay^{-b}+5x^{-a}y^b)(3x^ay^b-5x^{-a}y^{-b})$.

**11.** $\left(a^x-\frac{1}{2}-a^{-x}\right)^2$. **12.** $(x^{\frac{1}{a}}-x^{-\frac{1}{a}}+x)^2$.

**13.** $\{(a+b)^{\frac{1}{2}}+(a-b)^{\frac{1}{2}}\}^2$. **14.** $\{(a+b)^{\frac{1}{2}}-(a-b)^{-\frac{1}{2}}\}^2$.

*Write down* the quotient of

**15.** $x-9a$ by $x^{\frac{1}{2}}+3a^{\frac{1}{2}}$. **16.** $x^{\frac{3}{2}}-27$ by $x^{\frac{1}{2}}-3$.

**17.** $a^{2x}-16$ by $a^x-4$. **18.** $x^{3a}+8$ by $x^a+2$.

**19.** $c^{2x}-c^{-x}$ by $c^x-c^{-\frac{x}{2}}$. **20.** $1-8a^{-3}$ by $1-2a^{-1}$.

**21.** $a^{4x}-x^6$ by $a^{2x}+x^3$. **22.** $x^{-4}-1$ by $x^{-1}+1$.

**23.** $x^{\frac{5}{3}}-1$ by $x^{\frac{1}{3}}-1$. **24.** $x^{5n}+32$ by $x^n+2$.

Find the value of

**25.** $(x+x^{\frac{1}{2}}-4)(x+x^{\frac{1}{2}}+4)$. **26.** $(2x^{\frac{1}{3}}+4+3x^{-\frac{1}{3}})(2x^{\frac{1}{3}}+4-3x^{-\frac{1}{3}})$.

**27.** $(2-x^{\frac{1}{3}}+x)(2+x^{\frac{1}{3}}+x)$. **28.** $(a^x+7+3a^{-x})(a^x-7-3a^{-x})$.

**29.** $\dfrac{a^{\frac{4}{3}}-8a^{\frac{1}{3}}b}{a^{\frac{2}{3}}+2\sqrt[3]{ab}+4b^{\frac{2}{3}}}$. **30.** $\dfrac{x-7x^{\frac{1}{2}}}{x-5\sqrt{x}-14}\div\left(1+\dfrac{2}{\sqrt{x}}\right)^{-1}$.

**31.** $\dfrac{x^{\frac{2}{3}}-4\sqrt[3]{x^{-2}}}{\sqrt[3]{x^2}+4+4x^{-\frac{2}{3}}}$. **32.** $\dfrac{a^{\frac{3}{2}}+ab}{ab-b^3}-\dfrac{\sqrt{a}}{\sqrt{a}-b}$.

# CHAPTER XXXI.

## ELEMENTARY SURDS.

249. DEFINITION. If the root of a quantity cannot be exactly obtained the root is called a **surd.**

Thus $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[5]{a^3}$, $\sqrt{a^2+b^2}$ are surds.

By reference to the preceding chapter it will be seen that these are only cases of fractional indices; for the above quantities might be written

$$2^{\frac{1}{2}},\ 5^{\frac{1}{3}},\ a^{\frac{3}{5}},\ (a^2+b^2)^{\frac{1}{2}}.$$

Since surds may always be expressed as quantities with fractional indices they are subject to the same laws of combination as other algebraical symbols.

250. A quantity may be expressed in a surd form without really being a surd. Thus $\sqrt[3]{x^6}$ or $x^{\frac{6}{3}}$, though apparently a surd, can be expressed in the equivalent form $x^2$.

251. A surd is sometimes called an **irrational quantity**; and quantities which are not surds are, for the sake of distinction, termed **rational quantities.**

252. In the case of numerical surds such as $\sqrt{2}$, $\sqrt[3]{5}$, ..., although the *exact* value can never be found, it can be determined to any degree of accuracy by carrying the process of evolution far enough.

Thus $\qquad \sqrt{5}=2{\cdot}236068\ldots\ldots\ldots\ldots;$

that is $\sqrt{5}$ lies between 2·23606 and 2·23607; and therefore the error in using either of these quantities instead of $\sqrt{5}$ is less than ·00001. By taking the root to a greater number of decimal places we can approximate still nearer to the true value.

It thus appears that it will never be *absolutely necessary* to introduce surds into numerical work, which can always be carried on to a certain degree of accuracy; but we shall in the present chapter prove laws for combination of surd quantities which will enable us to work with symbols such as $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[4]{a}$, ... with absolute accuracy so long as the symbols are kept in their surd form. Moreover it will be found that even where approximate numerical results are required, the work is considerably simplified and shortened by operating with surd symbols, and afterwards substituting numerical values, if necessary.

253. The *order* of a surd is indicated by the root symbol, or surd index. Thus $\sqrt[3]{x}$, $\sqrt[n]{a}$ are respectively surds of the third and $n^{\text{th}}$ orders.

The surds of the most frequent occurrence are those of the second order; they are sometimes called **quadratic surds.** Thus $\sqrt{3}$, $\sqrt{a}$, $\sqrt{x+y}$ are quadratic surds.

254. It will frequently be found convenient to express a rational quantity in a surd form.

A rational quantity may be expressed in the form of a surd of *any required order* by raising it to the power whose root the surd expresses, and prefixing the radical sign. Thus

$$5=\sqrt{25}=\sqrt[3]{125}=\sqrt[4]{625}=\sqrt[n]{5^n};$$
$$a+x=\sqrt{(a+x)^2}=\sqrt[6]{(a+x)^6}=\sqrt[n]{(a+x)^n}.$$

255. A surd of any order may be transformed into a surd of a different order.

*Examples.* (1) $\sqrt[3]{2}=2^{\frac{1}{3}}=2^{\frac{4}{12}}=\sqrt[12]{2^4}$.

(2) $\sqrt[p]{a}=a^{\frac{1}{p}}=a^{\frac{q}{pq}}=\sqrt[pq]{a^q}$.

256. Surds of different orders may be transformed into surds of the same order. This order may be *any* common multiple of each of the given orders, but it is usually most convenient to choose the *least* common multiple.

*Example.* Express $\sqrt[4]{a^3}$, $\sqrt[3]{b^2}$, $\sqrt[6]{a^5}$ as surds of the same lowest order.

The least common multiple of 4, 3, 6 is 12; and expressing the given surds as surds of the twelfth order they become $\sqrt[12]{a^9}$, $\sqrt[12]{b^8}$, $\sqrt[12]{a^{10}}$.

257. Surds of different orders may be arranged according to magnitude by transforming them into surds of the same order.

*Example.* Arrange $\sqrt{3}$, $\sqrt[3]{6}$, $\sqrt[4]{10}$ according to magnitude.

The least common multiple of 2, 3, 4 is 12; and, expressing the given surds as surds of the twelfth order, we have

$$\sqrt{3}=\sqrt[12]{3^6}=\sqrt[12]{729},$$
$$\sqrt[3]{6}=\sqrt[12]{6^4}=\sqrt[12]{1296},$$
$$\sqrt[4]{10}=\sqrt[12]{10^3}=\sqrt[12]{1000}.$$

Hence arranged in descending order of magnitude the surds are

$$\sqrt{6},\ \sqrt[3]{10},\ \sqrt[4]{3}.$$

## EXAMPLES XXXI. a.

Express as surds of the twelfth order with positive indices:

1. $x^{\frac{1}{3}}$.
2. $a^{-1}\div a^{-\frac{1}{2}}$.
3. $\sqrt[4]{ax^3}\times\sqrt[3]{a^{-1}x^{-2}}$.
4. $\dfrac{1}{a^{-\frac{3}{4}}}$.
5. $\dfrac{1}{\sqrt[8]{a^{-14}}}$.
6. $\sqrt[6]{\dfrac{1}{a^{-2}}}$.

Express as surds of the $n^{\text{th}}$ order with positive indices:

7. $\sqrt[3]{x^2}$.
8. $x^a$.
9. $a^{\frac{1}{2}}$.
10. $\sqrt{a^{-\frac{1}{n}}}$.
11. $\sqrt[3]{x^ny^{\frac{1}{n}}}$.
12. $\dfrac{1}{a^{-1}}$.
13. $\dfrac{x^{-\frac{1}{2}}}{y^2}$.
14. $\dfrac{a^{\frac{1}{2}}}{x^{-n}}$.

Express as surds of the same lowest order:

15. $\sqrt{a}$, $\sqrt[9]{a^5}$.
16. $\sqrt[5]{a^3}$, $\sqrt{a}$.
17. $\sqrt[8]{x^3}$, $\sqrt[9]{x^6}$, $\sqrt[20]{x^5}$.
18. $\sqrt[16]{x^4}$, $\sqrt[12]{x^{10}}$.
19. $\sqrt[21]{a^3b^4}$, $\sqrt[7]{ab}$.
20. $\sqrt{ax^2}$, $\sqrt[30]{a^9x^6}$.
21. $\sqrt{5}$, $\sqrt[3]{11}$, $\sqrt[6]{13}$.
22. $\sqrt[4]{8}$, $\sqrt{3}$, $\sqrt[8]{6}$.
23. $\sqrt[3]{2}$, $\sqrt[9]{8}$, $\sqrt[6]{4}$.

258. The root of any expression is equal to the product of the roots of the separate factors of the expression.

For
$$\sqrt[n]{ab}=(ab)^{\frac{1}{n}}$$
$$=a^{\frac{1}{n}}b^{\frac{1}{n}}, \qquad \text{[Art. 244.]}$$
$$=\sqrt[n]{a}\,.\,\sqrt[n]{b}.$$

Similarly, $\sqrt[n]{abc}=\sqrt[n]{a}\,.\,\sqrt[n]{b}\,.\,\sqrt[n]{c}$;
and so for any number of factors.

*Examples.* (1) $\sqrt[4]{15}=\sqrt[4]{3}\,.\,\sqrt[4]{5}$.

(2) $\sqrt[3]{a^6b}=\sqrt[3]{a^6}\,.\,\sqrt[3]{b}=a^2\sqrt[3]{b}$.

(3) $\sqrt{50}=\sqrt{25}\,.\,\sqrt{2}=5\sqrt{2}$.

Hence it appears that a surd may sometimes be expressed as the product of a rational quantity and a surd; when so reduced the surd is said to be in its *simplest form.*

Thus the simplest form of $\sqrt{128}$ is $8\sqrt{2}$.

Conversely, the coefficient of a surd may be brought under the radical sign by first reducing it to the form of a surd, and then multiplying the surds together.

*Examples.* (1) $7\sqrt{5}=\sqrt{49}\,.\,\sqrt{5}=\sqrt{245}$.

(2) $a\sqrt[3]{b}=\sqrt[3]{a^3}\,.\,\sqrt[3]{b}=\sqrt[3]{a^3b}$.

When so reduced a surd is said to be an *entire surd.*

259. When surds have, or can be reduced to have, the same irrational factor, they are said to be *like;* otherwise, they are said to be *unlike.* Thus

$$5\sqrt{3},\ 2\sqrt{3},\ \frac{1}{5}\sqrt{3} \text{ are like surds.}$$

But $3\sqrt{2}$ and $2\sqrt{3}$ are unlike surds.

Again, $3\sqrt{20},\ 4\sqrt{5},\ \sqrt{\frac{1}{5}}$ are like surds;

for $3\sqrt{20}=3\sqrt{4}\,.\,\sqrt{5}=3\,.\,2\sqrt{5}=6\sqrt{5}$;

and $\sqrt{\frac{1}{5}}=\sqrt{\frac{5}{25}}=\frac{1}{5}\sqrt{5}$.

260. In finding the sum of a number of like surds we reduce them to their simplest form, and prefix to their common irrational part the sum of the coefficients.

*Example* 1. The sum of $3\sqrt{20},\ 4\sqrt{5},\ \frac{1}{\sqrt{5}}$

$$=6\sqrt{5}+4\sqrt{5}+\frac{1}{5}\sqrt{5}$$

$$=\frac{51}{5}\sqrt{5}.$$

*Example* 2. The sum of $x\sqrt[3]{8x^3a}+y\sqrt[3]{-y^3a}-z\sqrt[3]{z^3a}$

$$=x\,.\,2x\sqrt[3]{a}+y\,(-y)\sqrt[3]{a}-z\,.\,z\sqrt[3]{a}$$

$$=(2x^2-y^2-z^2)\sqrt[3]{a}.$$

261. Unlike surds cannot be collected.

Thus the sum of $5\sqrt{2}$, $-2\sqrt{3}$ and $\sqrt{6}$ is $5\sqrt{2}-2\sqrt{3}+\sqrt{6}$, and cannot be further simplified.

## EXAMPLES XXXI. b.

Express in the simplest form:

1. $\sqrt{288}$. 2. $\sqrt{147}$. 3. $\sqrt[3]{256}$. 4. $\sqrt[3]{432}$.

5. $3\sqrt{150}$. 6. $2\sqrt{720}$. 7. $5\sqrt{245}$. 8. $\sqrt[3]{1029}$.

9. $\sqrt[4]{3125}$. 10. $\sqrt[3]{-2187}$. 11. $\sqrt{36a^3}$. 12. $\sqrt{27a^3b^5}$.

**13.** $\sqrt[3]{-108x^4y^3}$.    **14.** $\sqrt[n]{x^{3n}y^{2n+5}}$.    **15.** $\sqrt[p]{x^{a+p}y^{2p}}$.

**16.** $\sqrt{a^3+2a^2b+ab^2}$.    **17.** $\sqrt[3]{8x^4y-24x^3y^2+24x^2y^3-8xy^4}$.

Express as entire surds:

**18.** $11\sqrt{2}$.    **19.** $14\sqrt{5}$.    **20.** $6\sqrt[3]{4}$.    **21.** $5\sqrt[3]{6}$.

**22.** $\frac{4}{11}\sqrt{\frac{77}{8}}$.    **23.** $\frac{3ab}{2c}\sqrt{\frac{20c^2}{9a^2b}}$.    **24.** $\frac{3x}{y}\sqrt{\frac{a^2y^3}{x^2}}$.

**25.** $\frac{a}{x^2}\sqrt{\frac{3x^3}{a}}$.    **26.** $\frac{2a}{3x}\sqrt[3]{\frac{27x^4}{a^2}}$.    **27.** $\frac{2a}{b}\sqrt[4]{\frac{b^4}{8a^3}}$.

**28.** $a\sqrt[n]{\frac{b^2}{a^{n-2}}}$.    **29.** $\frac{a}{b}\sqrt[p]{\frac{b^{p+1}}{a^{p-1}}}$.    **30.** $\frac{y}{x^n}\sqrt{\frac{x^{2n+1}}{y^3}}$.

**31.** $(x+y)\sqrt{\frac{x-y}{x+y}}$.    **32.** $\frac{ax}{a-x}\sqrt{\frac{a^2-x^2}{a^2x^2}}$.

Find the value of

**33.** $3\sqrt{45}-\sqrt{20}+7\sqrt{5}$.    **34.** $4\sqrt{63}+5\sqrt{7}-8\sqrt{28}$.

**35.** $\sqrt{44}-5\sqrt{176}+2\sqrt{99}$.    **36.** $2\sqrt{363}-5\sqrt{243}+\sqrt{192}$.

**37.** $2\sqrt[3]{189}+3\sqrt[3]{875}-7\sqrt[3]{56}$.    **38.** $5\sqrt[3]{81}-7\sqrt[3]{192}+4\sqrt[3]{648}$.

**39.** $3\sqrt[4]{162}-7\sqrt[4]{32}+\sqrt[4]{1250}$.    **40.** $5\sqrt[3]{-54}-2\sqrt[3]{-16}+4\sqrt[3]{686}$.

**41.** $4\sqrt{128}+4\sqrt{75}-5\sqrt{162}$.    **42.** $5\sqrt{24}-2\sqrt{54}-\sqrt{6}$.

**43.** $\sqrt{252}-\sqrt{294}-48\sqrt{\frac{1}{6}}$.    **44.** $3\sqrt{147}-\frac{7}{3}\sqrt{\frac{1}{3}}-\sqrt{\frac{1}{27}}$.

262. *To multiply two surds of the same order: multiply separately the rational factors and the irrational factors.*

For
$$a\sqrt[n]{x}\times b\sqrt[n]{y}=ax^{\frac{1}{n}}\times by^{\frac{1}{n}}$$
$$=abx^{\frac{1}{n}}y^{\frac{1}{n}}$$
$$=ab(xy)^{\frac{1}{n}}$$
$$=ab\sqrt[n]{xy}.$$

*Examples.* (1) $5\sqrt{3}\times 3\sqrt{7}=15\sqrt{21}$.

(2) $2\sqrt{x}\times 3\sqrt{x}=6x$.

(3) $\sqrt[4]{a+b}\times\sqrt[4]{a-b}=\sqrt[4]{(a+b)(a-b)}=\sqrt[4]{a^2-b^2}$.

263. If the surds are not in their simplest form, it will save labour to reduce them to this form before multiplication.

*Example.* The product of $5\sqrt{32}$, $\sqrt{48}$, $2\sqrt{54}$

$$=5\,.\,4\sqrt{2}\times 4\sqrt{3}\times 2\,.\,3\sqrt{6}=480\,.\,\sqrt{2}\,.\,\sqrt{3}\,.\,\sqrt{6}=480\times 6=2880.$$

264. *To multiply surds which are not of the same order: reduce them to equivalent surds of the same order, and proceed as before.*

*Example.* Multiply $5\sqrt[3]{2}$ by $2\sqrt{5}$.

The product $=5\sqrt[6]{2^2}\times 2\sqrt[6]{5^3}=10\sqrt[6]{2^4\times 5^3}=10\sqrt[6]{500}$.

265. Suppose it is required to find the numerical value of the quotient when $\sqrt{5}$ is divided by $\sqrt{7}$.

At first sight it would seem that we must find the square root of 5, which is 2·236..., and then the square root of 7, which is 2·645..., and finally divide 2·236... by 2·645...; three troublesome operations.

But we may avoid much of this labour by multiplying both numerator and denominator by $\sqrt{7}$, so as to make the denominator a rational quantity. Thus

$$\frac{\sqrt{5}}{\sqrt{7}}=\frac{\sqrt{5}}{\sqrt{7}}\times\frac{\sqrt{7}}{\sqrt{7}}=\frac{\sqrt{5\times 7}}{7}=\frac{\sqrt{35}}{7}.$$

Now $$\sqrt{35}=5{\cdot}916\ldots$$

$$\therefore\ \frac{\sqrt{5}}{\sqrt{7}}=\frac{5{\cdot}916\ldots}{7}={\cdot}845\ldots$$

266. The great utility of this artifice in calculating the numerical value of surd fractions suggests its convenience in the case of *all* surd fractions, even where numerical values are not required. Thus it is usual to simplify $\dfrac{a\sqrt{b}}{\sqrt{c}}$ as follows:

$$\frac{a\sqrt{b}}{\sqrt{c}}=\frac{a\sqrt{b}\times\sqrt{c}}{\sqrt{c}\times\sqrt{c}}=\frac{a\sqrt{bc}}{c}.$$

The process by which surds are removed from the denominator of any fraction is known as **rationalising the denominator.** It is effected by multiplying both numerator and denominator by any factor which renders the denominator rational. We shall return to this point in Art. 270.

267. The quotient of one surd by another may be found by expressing the result as a fraction, and rationalising the denominator.

*Example* 1. Divide $4\sqrt{75}$ by $25\sqrt{56}$.

$$\text{The quotient}=\frac{4\sqrt{75}}{25\sqrt{56}}=\frac{4\times5\sqrt{3}}{25\times2\sqrt{14}}=\frac{2\sqrt{3}}{5\sqrt{14}}$$

$$=\frac{2\sqrt{3}\times\sqrt{14}}{5\sqrt{14}\times\sqrt{14}}=\frac{2\sqrt{42}}{5\times14}=\frac{\sqrt{42}}{35}.$$

*Example* 2. $$\frac{\sqrt[3]{b}}{\sqrt[3]{c^2}}=\frac{\sqrt[3]{b}\times\sqrt[3]{c}}{\sqrt[3]{c^2}\times\sqrt[3]{c}}=\frac{\sqrt[3]{bc}}{\sqrt[3]{c^3}}=\frac{\sqrt[3]{bc}}{c}.$$

## EXAMPLES XXXI. c.

Find the value of

1. $2\sqrt{14}\times\sqrt{21}$. 2. $3\sqrt{8}\times\sqrt{6}$. 3. $5\sqrt{a}\times2\sqrt{3}$.

4. $2\sqrt{15}\times3\sqrt{5}$. 5. $8\sqrt{12}\times3\sqrt{24}$. 6. $\sqrt[3]{x+2}\times\sqrt[3]{x-2}$.

7. $21\sqrt{384}\div8\sqrt{98}$. 8. $5\sqrt{27}\div3\sqrt{24}$. 9. $-13\sqrt{125}\div5\sqrt{65}$.

10. $\sqrt[3]{168}\times\sqrt[3]{147}$. 11. $5\sqrt[3]{128}\times2\sqrt[3]{432}$. 12. $6\sqrt{14}\div2\sqrt{21}$.

13. $a\sqrt{b^3}\times b^2\sqrt{a}$. 14. $\dfrac{3\sqrt{11}}{2\sqrt{98}}\div\dfrac{5}{7\sqrt{22}}$. 15. $\dfrac{3\sqrt{48}}{5\sqrt{112}}\div\dfrac{6\sqrt{84}}{\sqrt{392}}$.

16. $\dfrac{3}{x}\sqrt{\dfrac{a^2}{x}}\times\dfrac{4}{3}\sqrt{\dfrac{x^3}{2a^4}}$. 17. $\dfrac{3}{a-b}\sqrt{\dfrac{2x}{a-b}}\div\sqrt{\dfrac{18x^3}{(a-b)^5}}$.

Given $\sqrt{2}=1{\cdot}41421$, $\sqrt{3}=1{\cdot}73205$, $\sqrt{5}=2{\cdot}23607$, $\sqrt{6}=2{\cdot}44949$, $\sqrt{7}=2{\cdot}64575$: find to four places of decimals the numerical value of

18. $\dfrac{14}{\sqrt{2}}$. 19. $\dfrac{25}{\sqrt{5}}$. 20. $\dfrac{10}{\sqrt{7}}$. 21. $\dfrac{48}{\sqrt{6}}$.

22. $\dfrac{60}{\sqrt{5}}$. 23. $144\div\sqrt{6}$. 24. $\sqrt{2}\div\sqrt{3}$. 25. $\dfrac{1}{2\sqrt{3}}$.

26. $\dfrac{1}{\sqrt{500}}$. 27. $\dfrac{4}{\sqrt{243}}$. 28. $\dfrac{25}{\sqrt{252}}$. 29. $\sqrt{\dfrac{256}{1575}}$.

268. Hitherto we have confined our attention to **simple surds**, such as $\sqrt[4]{5}$, $\sqrt[3]{a}$, $\sqrt{x+y}$. An expression involving two or more simple surds is called a **compound surd**; thus $2\sqrt{a}-3\sqrt{b}$; $\sqrt[3]{a}+\sqrt[4]{b}$ are compound surds.

**269.** The multiplication of compound surds is performed like the multiplication of compound algebraical expressions.

*Example* 1. Multiply $2\sqrt{x}-5$ by $3\sqrt{x}$.

$$\begin{aligned}\text{The product} &= 3\sqrt{x}\,(2\sqrt{x}-5)\\ &= 6x-15\sqrt{x}.\end{aligned}$$

*Example* 2. Multiply $2\sqrt{5}+3\sqrt{x}$ by $\sqrt{5}-\sqrt{x}$.

$$\begin{aligned}\text{The product} &= (2\sqrt{5}+3\sqrt{x})\,(\sqrt{5}-\sqrt{x})\\ &= 2\sqrt{5}\,.\,\sqrt{5}+3\sqrt{5}\,.\,\sqrt{x}-2\sqrt{5}\,.\,\sqrt{x}-3\sqrt{x}\,.\,\sqrt{x}\\ &= 10-3x+\sqrt{5x}.\end{aligned}$$

*Example* 3. Find the square of $2\sqrt{x}+\sqrt{7-4x}$.

$$\begin{aligned}(2\sqrt{x}+\sqrt{7-4x})^2 &= (2\sqrt{x})^2+(\sqrt{7-4x})^2+4\sqrt{x}\,.\,\sqrt{7-4x}\\ &= 4x+7-4x+4\sqrt{7x-4x^2}\\ &= 7+4\sqrt{7x-4x^2}.\end{aligned}$$

## EXAMPLES XXXI. d.

Find the value of

1. $(3\sqrt{x}-5)\times 2\sqrt{x}$.
2. $(\sqrt{x}-\sqrt{a})\times 2\sqrt{x}$.
3. $(\sqrt{a}+\sqrt{b})\times\sqrt{ab}$.
4. $(\sqrt{x+y}-1)\times\sqrt{x+y}$.
5. $(2\sqrt{3}+3\sqrt{2})^2$.
6. $(\sqrt{7}+5\sqrt{3})(2\sqrt{7}-4\sqrt{3})$.
7. $(3\sqrt{5}-4\sqrt{2})(2\sqrt{5}+3\sqrt{2})$.
8. $(3\sqrt{a}-2\sqrt{x})(2\sqrt{a}+3\sqrt{x})$.
9. $(\sqrt{x}+\sqrt{x-1})\times\sqrt{x-1}$.
10. $(\sqrt{x+a}-\sqrt{x-a})\times\sqrt{x+a}$.
11. $(\sqrt{a+x}-2\sqrt{a})^2$.
12. $(2\sqrt{a}-\sqrt{1+4a})^2$.
13. $(\sqrt{a+x}-\sqrt{a-x})^2$.
14. $(\sqrt{a+x}-2)(\sqrt{a+x}-1)$.
15. $(\sqrt{2}+\sqrt{3}-\sqrt{5})(\sqrt{2}+\sqrt{3}+\sqrt{5})$.
16. $(\sqrt{5}+3\sqrt{2}+\sqrt{7})(\sqrt{5}+3\sqrt{2}-\sqrt{7})$.

Write down the square of

17. $\sqrt{2x+a}-\sqrt{2x-a}$.
18. $\sqrt{x^2-2y^2}+\sqrt{x^2+2y^2}$.
19. $\sqrt{m+n}+\sqrt{m-n}$.
20. $3\sqrt{a^2+b^2}-2\sqrt{a^2-b^2}$.
21. $3x\sqrt{2}-3\sqrt{7-2x^2}$.
22. $\sqrt{4x^2+1}-\sqrt{4x^2-1}$.

270. One case of the multiplication of compound surds deserves careful attention. For if we multiply together the sum and the difference of any two quadratic surds we obtain a rational product.

*Examples.* (1) $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=(\sqrt{a})^2-(\sqrt{b})^2=a-b.$

(2) $(3\sqrt{5}+4\sqrt{3})(3\sqrt{5}-4\sqrt{3})=(3\sqrt{5})^2-(4\sqrt{3})^2=45-48=-3.$

Similarly, $(4-\sqrt{a+b})(4+\sqrt{a+b})=(4)^2-(\sqrt{a+b})^2=16-a-b.$

271. DEFINITION. When two binomial quadratic surds differ only in the sign which connects their terms they are said to be **conjugate.**

Thus $3\sqrt{7}+5\sqrt{11}$ is conjugate to $3\sqrt{7}-5\sqrt{11}$.

Similarly, $a-\sqrt{a^2-x^2}$ is conjugate to $a+\sqrt{a^2-x^2}$.

The product of two conjugate surds is rational. [Art. 270.]

*Example.* $(3\sqrt{a}+\sqrt{x-9a})(3\sqrt{a}-\sqrt{x-9a})$

$$=(3\sqrt{a})^2-(\sqrt{x-9a})^2=9a-(x-9a)=18a-x.$$

272. The only case of the division of compound surds which we shall here consider is that in which the divisor is a binomial quadratic surd. If we express the division by means of a fraction, we can always rationalise the denominator by multiplying numerator and denominator by the surd which is conjugate to the divisor.

*Example* 1. Divide $4+3\sqrt{2}$ by $5-3\sqrt{2}$.

$$\text{The quotient }=\frac{4+3\sqrt{2}}{5-3\sqrt{2}}=\frac{4+3\sqrt{2}}{5-3\sqrt{2}}\times\frac{5+3\sqrt{2}}{5+3\sqrt{2}}$$

$$=\frac{20+18+12\sqrt{2}+15\sqrt{2}}{25-18}=\frac{38+27\sqrt{2}}{7}.$$

*Example* 2. Rationalise the denominator of $\dfrac{b^2}{\sqrt{a^2+b^2}+a}$.

$$\text{The expression }=\frac{b^2}{\sqrt{a^2+b^2}+a}\times\frac{\sqrt{a^2+b^2}-a}{\sqrt{a^2+b^2}-a}$$

$$=\frac{b^2\{\sqrt{a^2+b^2}-a\}}{(a^2+b^2)-a^2}$$

$$=\sqrt{a^2+b^2}-a.$$

*Example* 3. Divide $\dfrac{\sqrt{3}+\sqrt{2}}{2-\sqrt{3}}$ by $\dfrac{7+4\sqrt{3}}{\sqrt{3}-\sqrt{2}}$.

$$\text{The quotient } =\frac{\sqrt{3}\times\sqrt{2}}{2-\sqrt{3}}\times\frac{\sqrt{3}-\sqrt{2}}{7+4\sqrt{3}}$$

$$=\frac{(\sqrt{3})^2-(\sqrt{2})^2}{14-12+8\sqrt{3}-7\sqrt{3}}$$

$$=\frac{1}{2+\sqrt{3}}$$

$$=2-\sqrt{3}, \text{ on rationalising.}$$

*Example* 4. Given $\sqrt{5}=2{\cdot}236068$, find the value of $\dfrac{87}{7-2\sqrt{5}}$.

Rationalising the denominator,

$$\frac{87}{7-2\sqrt{5}}=\frac{87(7+2\sqrt{5})}{49-20}$$

$$=3(7+2\sqrt{5})$$

$$=34{\cdot}416408.$$

It will be seen that by rationalising the denominator we have avoided the use of a divisor consisting of 7 figures.

## EXAMPLES XXXI. e.

Find the value of

**1.** $(9\sqrt{2}-7)(9\sqrt{2}+7)$.

**2.** $(3+5\sqrt{7})(3-5\sqrt{7})$.

**3.** $(5\sqrt{8}-2\sqrt{7})(5\sqrt{8}+2\sqrt{7})$.

**4.** $(2\sqrt{11}+5\sqrt{2})(2\sqrt{11}-5\sqrt{2})$.

**5.** $(\sqrt{a}+2\sqrt{b})(\sqrt{a}-2\sqrt{b})$.

**6.** $(3c-2\sqrt{x})(3c+2\sqrt{x})$.

**7.** $(\sqrt{a+x}-\sqrt{a})(\sqrt{a+x}+\sqrt{a})$.

**8.** $(\sqrt{2p+3q}-2\sqrt{q})(\sqrt{2p+3q}+2\sqrt{q})$.

**9.** $(\sqrt{a+x}+\sqrt{a-x})(\sqrt{a+x}-\sqrt{a-x})$.

**10.** $(5\sqrt{x^2-3y^2}+7a)(5\sqrt{x^2-3y^2}-7a)$.

**11.** $29\div(11+3\sqrt{7})$.

**12.** $17\div(3\sqrt{7}+2\sqrt{3})$.

**13.** $(3\sqrt{2}-1)\div(3\sqrt{2}+1)$.

**14.** $(2\sqrt{3}+7\sqrt{2})\div(5\sqrt{3}-4\sqrt{2})$.

**15.** $(2x-\sqrt{xy})\div(2\sqrt{xy}-y)$.

**16.** $(3+\sqrt{5})(\sqrt{5}-2)\div(5-\sqrt{5})$.

**17.** $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{x}}\div\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{x}}$.

**18.** $\dfrac{2\sqrt{15}+8}{5+\sqrt{15}}\div\dfrac{8\sqrt{3}-6\sqrt{5}}{5\sqrt{3}-3\sqrt{5}}$.

Rationalise the denominator of

**19.** $\dfrac{25\sqrt{3}-4\sqrt{2}}{7\sqrt{3}-5\sqrt{2}}$.

**20.** $\dfrac{10\sqrt{6}-2\sqrt{7}}{3\sqrt{6}+2\sqrt{7}}$.

**21.** $\dfrac{\sqrt{7}+\sqrt{2}}{9+2\sqrt{14}}$.

22. $\dfrac{2\sqrt{3}+3\sqrt{2}}{5+2\sqrt{6}}$. 23. $\dfrac{y^2}{x+\sqrt{x^2-y^2}}$. 24. $\dfrac{x^2}{\sqrt{x^2+a^2}+a}$.

25. $\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$. 26. $\dfrac{2\sqrt{a+b}+3\sqrt{a-b}}{2\sqrt{a+b}-\sqrt{a-b}}$.

27. $\dfrac{\sqrt{9+x^2}-3}{\sqrt{9+x^2}+3}$. 28. $\dfrac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}$.

Given $\sqrt{2}=1{\cdot}41421$, $\sqrt{3}=1{\cdot}73205$, $\sqrt{5}=2{\cdot}23607$: find to four places of decimals the value of

29. $\dfrac{1}{2+\sqrt{3}}$. 30. $\dfrac{3+\sqrt{5}}{\sqrt{5}-2}$. 31. $\dfrac{\sqrt{5}+\sqrt{3}}{4+\sqrt{15}}$. 32. $\dfrac{\sqrt{5}-2}{9-4\sqrt{5}}$.

33. $\dfrac{7\sqrt{5}+15}{\sqrt{5}-1}\times\dfrac{\sqrt{5}-2}{3+\sqrt{5}}$. 34. $(2-\sqrt{3})(7-4\sqrt{3})\div(3\sqrt{3}-5)$.

273. *The square root of a rational quantity cannot be partly rational and partly a quadratic surd.*

If possible let $\sqrt{n}=a+\sqrt{m}$;

then by squaring, $n=a^2+m+2a\sqrt{m}$;

$$\therefore \quad \sqrt{m}=\frac{n-a^2-m}{2a}.$$

that is a surd is equal to a rational quantity; which is impossible.

274. *If* $x+\sqrt{y}=a+\sqrt{b}$, *then will* $x=a$ *and* $y=b$.

For if $x$ is not equal to $a$, let $x=a+m$; then

$$a+m+\sqrt{y}=a+\sqrt{b};$$

that is, $\sqrt{b}=m+\sqrt{y}$;

which is impossible. [Art. 273.]

Therefore $x=a$,

and consequently, $y=b$,

If therefore $x+\sqrt{y}=a+\sqrt{b}$,

we must also have $x-\sqrt{y}=a-\sqrt{b}$.

275. It appears from the preceding article that in any equation of the form

$$X+\sqrt{Y}=A+\sqrt{B}\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

we may equate the rational parts on each side, and also the irrational parts; so that the equation (1) is really equivalent to *two* independent equations, $X=A$, and $Y=B$.

276. *If* $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$ *then will* $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

For by squaring, we obtain

$$a+\sqrt{b}=x+2\sqrt{xy}+y;$$

$$\therefore \quad a=x+y, \ \sqrt{b}=2\sqrt{xy}. \qquad \text{[Art. 275.]}$$

Hence
$$a-\sqrt{b}=x-2\sqrt{xy}+y,$$
and
$$\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}.$$

277. *To find the square root of* $a+\sqrt{b}$.

Suppose
$$\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y};$$
then as in the last article,

$$x+y=a \ \ldots\ldots\ldots\ldots(1),$$
$$2\sqrt{xy}=\sqrt{b} \ \ldots\ldots\ldots\ldots(2).$$

$$\therefore \quad (x-y)^2=(x+y)^2-4xy$$
$$=a^2-b \qquad \text{from (1) and (2).}$$
$$\therefore \quad x-y=\sqrt{a^2-b}.$$

Combining this with (1) we find

$$x=\frac{a+\sqrt{a^2-b}}{2}, \text{ and } y=\frac{a-\sqrt{a^2-b}}{2}.$$

$$\therefore \quad \sqrt{a+\sqrt{b}}=\sqrt{\frac{a+\sqrt{(a^2-b)}}{2}}+\sqrt{\frac{a-\sqrt{(a^2-b)}}{2}}.$$

278. From the values just found for $x$ and $y$, it appears that each of them is itself a compound surd unless $a^2-b$ is a perfect square. Hence the method of Art. 277 for finding the square root of $a+\sqrt{b}$ is of no practical utility except when $a^2-b$ is a perfect square.

*Example.* Find the square root of $16+2\sqrt{55}$.

Assume
$$\sqrt{16+2\sqrt{55}}=\sqrt{x}+\sqrt{y}.$$
Then
$$16+2\sqrt{55}=x+2\sqrt{xy}+y;$$
$$\therefore \ x+y=16 \ \ldots\ldots\ldots\ldots (1),$$
$$2\sqrt{xy}=2\sqrt{55} \ \ldots\ldots\ldots\ldots (2).$$
$$\therefore \ (x-y)^2=(x+y)^2-4xy$$
$$=16^2-4\times 55 \ \ldots\ldots \text{ by (1) and (2).}$$
$$=4\times 9.$$
$$\therefore \ x-y=\pm 6 \ \ldots\ldots\ldots\ldots (3).$$

From (1) and (3) we obtain

$$x=11, \text{ or } 5, \text{ and } y=5, \text{ or } 11.$$

That is, the required square root is $\sqrt{11}+\sqrt{5}$.

In the same way we may shew that

$$\sqrt{16-2\sqrt{55}}=\sqrt{11}-\sqrt{5}.$$

NOTE. Since every quantity has two square roots, equal in magnitude but opposite in sign, strictly speaking we should have

$$\text{the square root of } 16+2\sqrt{55}=\pm(\sqrt{11}+\sqrt{5}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; 16-2\sqrt{55}=\pm(\sqrt{11}-\sqrt{5}).$$

However it is usually sufficient to take the positive value of the square root, so that in assuming $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$ it is understood that $x$ is greater than $y$. With this proviso it will be unnecessary in any numerical example to use the double sign at the stage of work corresponding to equation (3) of the last example.

279. When the binomial whose square root we are seeking consists of *two* quadratic surds, we proceed as explained in the following example.

*Example.* Find the square root of $\sqrt{175}-\sqrt{147}$.

Since $\sqrt{175}-\sqrt{147}=\sqrt{7}\,(\sqrt{25}-\sqrt{21})=\sqrt{7}\,(5-\sqrt{21})$.

$$\therefore\ \sqrt{\sqrt{175}-\sqrt{147}}=\sqrt[4]{7}\,.\,\sqrt{5-\sqrt{21}}.$$

And, proceeding as in the last Article,

$$\sqrt{5-\sqrt{21}}=\sqrt{\frac{7}{2}}-\sqrt{\frac{3}{2}};$$

$$\therefore\ \sqrt{\sqrt{175}-\sqrt{147}}=\sqrt[4]{7}\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{3}{2}}\right).$$

280. The square root of a binomial surd may often be found by inspection.

*Example* 1. Find the square root of $11+2\sqrt{30}$.

We have only to find two quantities whose sum is 11, and whose product is 30; thus

$$11+2\sqrt{30}=6+5+2\sqrt{6\times5}$$
$$=(\sqrt{6}+\sqrt{5})^2.$$
$$\therefore\ \sqrt{11+2\sqrt{30}}=\sqrt{6}+\sqrt{5}.$$

*Example* 2. Find the square root of $53-12\sqrt{10}$.

First write the binomial so that the surd part has a coefficient 2; thus

$$53-12\sqrt{10}=53-2\sqrt{360}.$$

We have now to find two quantities whose sum is 53 and whose product is 360; these are 45 and 8;

hence
$$\begin{aligned} 53-12\sqrt{10} &= 45+8-2\sqrt{45\times 8} \\ &= (\sqrt{45}-\sqrt{8})^2; \end{aligned}$$
$$\begin{aligned} \therefore\ \sqrt{53-12\sqrt{10}} &= \sqrt{45}-\sqrt{8} \\ &= 3\sqrt{5}-2\sqrt{2}. \end{aligned}$$

## EXAMPLES XXXI. f.

Find the square root of each of the following binomial surds:

1. $7-2\sqrt{10}$.
2. $13+2\sqrt{30}$.
3. $8-2\sqrt{7}$.
4. $5+2\sqrt{6}$.
5. $75+12\sqrt{21}$.
6. $18-8\sqrt{5}$.
7. $41-24\sqrt{2}$.
8. $83+12\sqrt{35}$.
9. $47-4\sqrt{33}$.
10. $2\tfrac{1}{4}+\sqrt{5}$.
11. $4\tfrac{1}{3}-\dfrac{4}{3}\sqrt{3}$.
12. $16+5\sqrt{7}$.
13. $\sqrt{27}+2\sqrt{6}$.
14. $\sqrt{32}-\sqrt{24}$.
15. $3\sqrt{5}+\sqrt{40}$.

Find the fourth roots of the following binomial surds:

16. $17+12\sqrt{2}$.
17. $56+24\sqrt{5}$.
18. $\dfrac{3}{2}\sqrt{5}+3\tfrac{1}{2}$.
19. $14+8\sqrt{3}$.
20. $49-20\sqrt{6}$.
21. $248+32\sqrt{60}$.

Find, by inspection, the value of

22. $\sqrt{3-2\sqrt{2}}$.
23. $\sqrt{4+2\sqrt{3}}$.
24. $\sqrt{6-2\sqrt{5}}$.
25. $\sqrt{19+8\sqrt{3}}$.
26. $\sqrt{8+2\sqrt{15}}$.
27. $\sqrt{9-2\sqrt{14}}$.
28. $\sqrt{11+4\sqrt{6}}$.
29. $\sqrt{15-4\sqrt{14}}$.
30. $\sqrt{29+6\sqrt{22}}$.

### EQUATIONS INVOLVING SURDS.

281. Sometimes equations are proposed in which the unknown quantity appears under the radical sign. Such equations are very varied in character and often require special artifices for their solution. Here we shall only consider a few of the simpler cases, which can generally be solved by the following method. Bring to one side of the equation a single radical term by itself: on squaring both sides this radical will disappear. By repeating this process any remaining radicals can in turn be removed.

*Example* 1. Solve $2\sqrt{x}-\sqrt{4x-11}=1$.

Transposing, $$2\sqrt{x}-1=\sqrt{4x-11}.$$

Square both sides; then $4x-4\sqrt{x}+1=4x-11$,

$$4\sqrt{x}=12,$$
$$\sqrt{x}=3;$$
$$\therefore\ x=9.$$

*Example* 2. Solve $2+\sqrt[3]{x-5}=13$.

Transposing, $$\sqrt[3]{x-5}=11.$$

Here we must *cube* both sides; thus $x-5=1331$; whence $$x=1336.$$

*Example* 3. Solve $\sqrt{x+5}+\sqrt{3x+4}=\sqrt{12x+1}$.

Squaring both sides,

$$x+5+3x+4+2\sqrt{(x+5)(3x+4)}=12x+1.$$

Transposing and dividing by 2,

$$\sqrt{(x+5)(3x+4)}=4x-4 \text{ .................... } (1).$$

Squaring, $$(x+5)(3x+4)=16x^2-32x+16,$$

or $$13x^2-51x-4=0,$$
$$(x-4)(13x+1)=0;$$
$$\therefore\ x=4, \text{ or } -\frac{1}{13}.$$

If we proceed to verify the solution by substituting these values in the original equation, it will be found that it is satisfied by $x=4$, but not by $x=-\frac{1}{13}$. But this latter value will be found on trial to satisfy the given equation if we alter the sign of the second radical; thus $$\sqrt{x+5}-\sqrt{3x+4}=\sqrt{12x+1}.$$

On squaring this and reducing, we obtain

$$-\sqrt{(x+5)(3x+4)}=4x-4 \text{ .................... } (2);$$

and a comparison of (1) and (2) shews that in the next stage of the work *the same quadratic equation is obtained* in each case, the roots of which are 4 and $-\frac{1}{13}$, as already found.

From this it appears that when the solution of an equation requires that both sides should be squared, we cannot be certain without trial which of the values found for the unknown quantity will satisfy the original equation.

In order that all the values found by the solution of the equation may be applicable it will be necessary to take into account both signs of the radical in the given equation.

## EXAMPLES XXXI. g.

1. $\sqrt{x-5}=3$.
2. $\sqrt[3]{4x-7}=5$.
3. $7-\sqrt{x-4}=3$.
4. $13-\sqrt[3]{5x-4}=7$.
5. $\sqrt{5x-1}=2\sqrt{x+3}$.
6. $2\sqrt{3-7x}-3\sqrt{8x-12}=0$.
7. $2\sqrt[3]{5x-35}=5\sqrt[3]{2x-7}$.
8. $\sqrt{9x^2-11x-5}=3x-2$.
9. $\sqrt[4]{2x+11}=\sqrt{5}$.
10. $\sqrt{4x^2-7x+1}=2x-1\frac{4}{5}$.
11. $\sqrt{x+25}=1+\sqrt{x}$.
12. $\sqrt{8x+33}-3=2\sqrt{2x}$.
13. $\sqrt{x+3}+\sqrt{x}=5$.
14. $10-\sqrt{25+9x}=3\sqrt{x}$.
15. $\sqrt{x-4}+3=\sqrt{x+11}$.
16. $\sqrt{9x-8}=3\sqrt{x+4}-2$.
17. $\sqrt{4x+5}-\sqrt{x}=\sqrt{x+3}$.
18. $\sqrt{25x-29}-\sqrt{4x-11}=3\sqrt{x}$.
19. $\sqrt{8x+17}-\sqrt{2x}=\sqrt{2x+9}$.
20. $\sqrt{3x-11}+\sqrt{3x}=\sqrt{12x-23}$.
21. $\sqrt{12x-5}+\sqrt{3x-1}=\sqrt{27x-2}$.
22. $\sqrt{x+3}+\sqrt{x+8}-\sqrt{4x+21}=0$.
23. $\sqrt{x+2}+\sqrt{4x+1}-\sqrt{9x+7}=0$.
24. $\sqrt{x+4ab}=2a+\sqrt{x}$.
25. $\sqrt{x}+\sqrt{4a+x}=2\sqrt{b+x}$.

282. When radicals appear in a fractional form in an equation, we must clear of fractions in the ordinary way, combining the irrational factors by the rules already explained in this chapter.

*Example* 1. Solve $\dfrac{6\sqrt{x}-11}{3\sqrt{x}}=\dfrac{2\sqrt{x}+1}{\sqrt{x}+6}$.

Multiplying across, $6x+25\sqrt{x}-66=6x+3\sqrt{x}$,

$$25\sqrt{x}-3\sqrt{x}=66,$$

$$22\sqrt{x}=66,$$

$$\sqrt{x}=3,$$

$$x=9.$$

*Example* 2. Solve $\sqrt{9+2x}-\sqrt{2x}=\dfrac{5}{\sqrt{9+2x}}$.

Clearing of fractions,

$$9+2x-\sqrt{2x(9+2x)}=5,$$
$$4+2x=\sqrt{2x(9+2x)}.$$

Squaring, $$16+16x+4x^2=18x+4x^2,$$
$$16=2x,$$
$$x=8.$$

## EXAMPLES XXXI. h.

1. $\dfrac{6\sqrt{x}-21}{3\sqrt{x}-14}=\dfrac{8\sqrt{x}-11}{4\sqrt{x}-13}$.

2. $\dfrac{9\sqrt{x}-23}{3\sqrt{x}-8}=\dfrac{6\sqrt{x}-17}{2\sqrt{x}-6}$.

3. $\dfrac{\sqrt{x}+3}{\sqrt{x}-2}=\dfrac{3\sqrt{x}-5}{3\sqrt{x}-13}$.

4. $2-\dfrac{\sqrt{x}+3}{\sqrt{x}+2}=\dfrac{\sqrt{x}+9}{\sqrt{x}+7}$.

5. $\dfrac{2\sqrt{x}-1}{2\sqrt{x}+\frac{4}{3}}=\dfrac{\sqrt{x}-2}{\sqrt{x}-\frac{4}{3}}$.

6. $\dfrac{6\sqrt{x}-7}{\sqrt{x}-1}-5=\dfrac{7\sqrt{x}-26}{7\sqrt{x}-21}$.

7. $\dfrac{12\sqrt{x}-11}{4\sqrt{x}-4\frac{2}{3}}=\dfrac{6\sqrt{x}+5}{2\sqrt{x}+\frac{2}{3}}$.

8. $\sqrt{1+x}+\sqrt{x}=\dfrac{2}{\sqrt{1+x}}$.

9. $\sqrt{x-1}+\sqrt{x}=\dfrac{2}{\sqrt{x}}$.

10. $\sqrt{x}-\sqrt{x-8}=\dfrac{2}{\sqrt{x-8}}$.

11. $\sqrt{x+5}+\sqrt{x}=\dfrac{10}{\sqrt{x}}$.

12. $2\sqrt{x}-\sqrt{4x-3}=\dfrac{1}{\sqrt{4x-3}}$.

13. $3\sqrt{x}=\dfrac{8}{\sqrt{9x-32}}+\sqrt{9x-32}$.

14. $\sqrt{x}-7=\dfrac{1}{\sqrt{x}+7}$.

15. $(\sqrt{x}+11)(\sqrt{x}-11)+110=0$.

16. $2\sqrt{x}=\dfrac{12-6\sqrt{x}}{2\sqrt{x}-3}$.

17. $3\sqrt{x}-1=\dfrac{5}{3\sqrt{x}+7}+6$.

18. $\dfrac{x-1}{\sqrt{x}-1}=3+\dfrac{\sqrt{x}+1}{2}$.

19. $\dfrac{1}{1-x}+\dfrac{1}{\sqrt{x}+1}+\dfrac{1}{\sqrt{x}-1}=0$.

20. $2=\dfrac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}-\sqrt{2-x}}$.

21. $\dfrac{2x-3}{\sqrt{x-2}+1}=2\sqrt{x-2}-1$.

22. $\dfrac{2}{x-6+\sqrt{x}}+\dfrac{3}{\sqrt{x}-2}=\dfrac{4}{\sqrt{x}+3}$.

# CHAPTER XXXII.

## RATIO, PROPORTION, AND VARIATION.

283. DEFINITION. **Ratio** is the relation which one quantity bears to another of the *same* kind, the comparison being made by considering what multiple, part, or parts, one quantity is of the other.

The ratio of $A$ to $B$ is usually written $A : B$. The quantities $A$ and $B$ are called the *terms* of the ratio. The first term is called the **antecedent**, the second term the **consequent**.

284. To find what multiple or part $A$ is of $B$ we divide $A$ by $B$; hence the ratio $A : B$ may be measured by the fraction $\frac{A}{B}$, and we shall usually find it convenient to adopt this notation.

In order to compare two quantities they must be expressed in terms of the same unit. Thus the ratio of £2 to 15*s.* is measured by the fraction $\frac{2 \times 20}{15}$ or $\frac{8}{3}$.

NOTE. Since a ratio expresses the *number* of times that one quantity contains another, *every ratio is an abstract quantity.*

285. By Art. 151, $$\frac{a}{b} = \frac{ma}{mb};$$

and thus the ratio $a : b$ is equal to the ratio $ma : mb$; that is, *the value of a ratio remains unaltered if the antecedent and the consequent are multiplied or divided by the same quantity.*

286. Two or more ratios may be compared by reducing their equivalent fractions to a common denominator. Thus suppose $a : b$ and $x : y$ are two ratios. Now $\frac{a}{b} = \frac{ay}{by}$, and $\frac{x}{y} = \frac{bx}{by}$; hence the ratio $a : b$ is greater than, equal to, or less than the ratio $x : y$ according as $ay$ is greater than, equal to, or less than $bx$.

287. The ratio of two fractions can be expressed as a ratio of two integers. Thus the ratio $\frac{a}{b} : \frac{c}{d}$ is measured by the fraction $\dfrac{\frac{a}{b}}{\frac{c}{d}}$, or $\frac{ad}{bc}$; and is therefore equivalent to the ratio $ad : bc$.

288. If either, or both, of the terms of a ratio be a surd quantity, then no two integers can be found which will *exactly* measure their ratio. Thus the ratio $\sqrt{2} : 1$ cannot be exactly expressed by any two integers.

289. DEFINITION. If the ratio of any two quantities can be expressed exactly by the ratio of two integers the quantities are said to be **commensurable**; otherwise, they are said to be **incommensurable.**

Although we cannot find two integers which will exactly measure the ratio of two incommensurable quantities, we can always find two integers whose ratio differs from that required by as small a quantity as we please.

Thus $$\frac{\sqrt{5}}{4} = \frac{2{\cdot}236068\ldots}{4} = {\cdot}559017\ldots$$

and therefore $$\frac{\sqrt{5}}{4} \text{ is } > \frac{55901}{100000} \text{ and } < \frac{55902}{100000},$$

and it is evident that by carrying the decimals further, any degree of approximation may be arrived at.

290. DEFINITION. Ratios are *compounded* by multiplying together the fractions which denote them; or by multiplying together the antecedents for a new antecedent, and the consequents for a new consequent.

*Example.* The ratio compounded of the three ratios

$$2a : 3b,\ 6ab : 5c^2,\ c : a$$

is $$2a \times 6ab \times c : 3b \times 5c^2 \times a,$$

or $$4a : 5c.$$

291. DEFINITION. When the ratio $a : b$ is compounded with itself the resulting ratio is $a^2 : b^2$, and is called the **duplicate ratio** of $a : b$. Similarly $a^3 : b^3$ is called the **triplicate ratio** of $a : b$. Also $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is called the **subduplicate ratio** of $a : b$.

*Examples.* (1) The duplicate ratio of $2a : 3b$ is $4a^2 : 9b^2$.

(2) The subduplicate ratio of $49 : 25$ is $7 : 5$.

(3) The triplicate ratio of $2x : 1$ is $8x^3 : 1$.

292. DEFINITION. A ratio is said to be a ratio of *greater inequality*, of *less inequality*, or of *equality*, according as the antecedent is *greater than*, *less than*, or *equal to* the consequent.

293. If to each term of the ratio $8 : 3$ we add 4, a new ratio $12 : 7$ is obtained, and we see that it is less than the former because $\frac{12}{7}$ is clearly less than $\frac{8}{3}$.

This is a particular case of a more general proposition which we shall now prove.

*A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same quantity to both its terms.*

Let $\frac{a}{b}$ be the ratio, and let $\frac{a+x}{b+x}$ be the new ratio formed by adding $x$ to both its terms.

Now
$$\frac{a}{b} - \frac{a+x}{b+x} = \frac{ax-bx}{b(b+x)}$$
$$= \frac{x(a-b)}{b(b+x)};$$

and $a-b$ is positive or negative according as $a$ is greater or less than $b$.

Hence if $a$ is $>b$, $\frac{a}{b}$ is $> \frac{a+x}{b+x}$;

and if $a$ is $<b$, $\frac{a}{b}$ is $< \frac{a+x}{b+x}$.

which proves the proposition.

Similarly it can be proved that *a ratio of greater inequality is increased, and a ratio of less inequality is diminished, by taking the same quantity from both its terms.*

294. When two or more ratios are equal, many useful propositions may be proved by introducing a single symbol to denote each of the equal ratios.

The proof of the following important theorem will illustrate the method of procedure.

*If* $$\frac{\mathrm{a}}{\mathrm{b}}=\frac{\mathrm{c}}{\mathrm{d}}=\frac{\mathrm{e}}{\mathrm{f}}=\ldots\ldots,$$

*each of these ratios* $=\left(\dfrac{\mathrm{pa}^n+\mathrm{qc}^n+\mathrm{re}^n+\ldots}{\mathrm{pb}^n+\mathrm{qd}^n+\mathrm{rf}^n+\ldots}\right)^{\frac{1}{n}}$,

*where* p, q, r, n *are any quantities whatever.*

Let $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=k;$$

then $$a=bk,\ c=dk,\ e=fk,\ldots;$$

whence $$pa^n=pb^nk^n,\ qc^n=qd^nk^n,\ re^n=rf^nk^n,\ldots;$$

$$\therefore\ pa^n+qc^n+re^n+\ldots=k^n(pb^n+qd^n+rf^n+\ldots),$$

$$\frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}=k^n;$$

$$\therefore\ \left(\frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}\right)^{\frac{1}{n}}=k=\frac{a}{b}=\frac{c}{d}=\ldots.$$

By giving different values to $p$, $q$, $r$, $n$ many particular cases of this general proposition may be deduced; or they may be proved independently by using the same method. For instance,

if $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, each of these ratios $=\dfrac{a+c+e}{b+d+f}$;

a result which will frequently be found useful.

*Example* 1. If $\dfrac{x}{y}=\dfrac{3}{4}$ find the value of $\dfrac{5x-3y}{7x+2y}$,

$$\frac{5x-3y}{7x+2y}=\frac{\dfrac{5x}{y}-3}{\dfrac{7x}{y}+2}=\frac{\dfrac{15}{4}-3}{\dfrac{21}{4}+2}=\frac{3}{29}.$$

*Example* 2. Two numbers are in the ratio of 5 : 8. If 9 be added to each they are in the ratio of 8 : 11. Find the numbers.

Let the numbers be denoted by $5x$ and $8x$.

Then $$\frac{5x+9}{8x+9}=\frac{8}{11};$$

$$\therefore\ x=3.$$

Hence the numbers are 15 and 24.

*Example* 3. If $A : B$ be in the duplicate ratio of $A+x : B+x$, prove that $x^2=AB$.

By the given condition, $\left(\dfrac{A+x}{B+x}\right)^2=\dfrac{A}{B}$;

$$\therefore\ B(A+x)^2=A(B+x)^2,$$
$$A^2B+2ABx+Bx^2=AB^2+2ABx+Ax^2,$$
$$x^2(A-B)=AB(A-B);$$
$$\therefore\ x^2=AB.$$

Since $A-B$ is, by supposition, not zero.

## EXAMPLES XXXII. a.

Find the ratio compounded of

1. The duplicate ratio of $4 : 3$, and the ratio $27 : 8$.
2. The ratio $32 : 27$, and the triplicate ratio of $3 : 4$.
3. The subduplicate ratio of $25 : 36$, and the ratio $6 : 25$.
4. The ratio $169 : 200$, and the duplicate ratio of $15 : 26$.
5. The triplicate ratio of $x : y$, and the ratio $2y^2 : 3x^2$.
6. The ratio $3a : 4b$, and the subduplicate ratio of $b^4 : a^4$.
7. If $x : y=5 : 7$, find the value of $x+y : y-x$.
8. If $\dfrac{x}{y}=3\tfrac{1}{3}$, find the value of $\dfrac{x-3y}{2x-5y}$.
9. If $b : a=2 : 5$, find the value of $2a-3b : 3b-a$.
10. If $\dfrac{a}{b}=\dfrac{3}{4}$, and $\dfrac{x}{y}=\dfrac{5}{7}$, find the value of $\dfrac{3ax-by}{4by-7ax}$.
11. If $7x-4y : 3x+y=5 : 13$, find the ratio $x : y$.
12. If $\dfrac{2a^2-3b^2}{a^2+b^2}=\dfrac{2}{41}$, find the ratio $a : b$.
13. If $2x : 3y$ be in the duplicate ratio of $2x-m : 3y-m$, prove that $m^2=6xy$.
14. If $P : Q$ be the subduplicate ratio of $P-x : Q-x$, prove that $x=\dfrac{PQ}{P+Q}$.
15. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, prove that each of these ratios is equal to
$$\sqrt[3]{\frac{2a^2c+3c^3e+4e^2c}{2b^2d+3d^3e+4f^2d}}.$$

16. Two numbers are in the ratio of $3 : 4$, and if 7 be subtracted from each the remainders are in the ratio of $2 : 3$. Find them.

17. What number must be taken from each term of the ratio $27 : 35$ that it may become $2 : 3$?

18. What number must be added to each term of the ratio $37 : 29$ that it may become $8 : 7$?

19. If $\frac{p}{b-c}=\frac{q}{c-a}=\frac{r}{a-b}$, shew that $p+q+r=0$.

20. If $\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a-b}$, shew that $x-y+z=0$.

21. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that the square root of

$$\frac{a^6b-2c^5e+3a^4c^3e^2}{b^7-2d^5f+3b^4cd^2e^2} \text{ is equal to } \frac{ace}{bdf}.$$

22. Prove that the ratio $la+mc+ne : lb+md+nf$ will be equal to each of the ratios $a : b, c : d, e : f$, if these be all equal; and that it will be intermediate in value between the greatest and least of these ratios if they be not all equal.

23. If $\frac{bx-ay}{cy-az}=\frac{cx-az}{by-ax}=\frac{z+y}{x+z}$, then will each of these fractions be equal to $\frac{x}{y}$, unless $b+c=0$.

24. If $\frac{2x-3y}{3z+y}=\frac{z-y}{z-x}=\frac{x+3z}{2y-3x}$, prove that each of these ratios is equal to $\frac{x}{y}$; hence shew that either $x=y$, or $z=x+y$.

## PROPORTION.

295. DEFINITION. When two ratios are equal, the four quantities composing them are said to be **proportionals.** Thus if $\frac{a}{b}=\frac{c}{d}$, then $a, b, c, d$ are proportionals. This is expressed by saying that $a$ is to $b$ as $c$ is to $d$, and the proportion is written

$$a : b :: c : d;$$

or

$$a : b = c : d.$$

The terms $a$ and $d$ are called the *extremes*, $b$ and $c$ the *means*.

296. *If four quantities are in proportion, the product of the extremes is equal to the product of the means.*

Let $a$, $b$, $c$, $d$ be the proportionals.

Then by definition $\frac{a}{b}=\frac{c}{d}$;

whence $ad=bc$.

Hence if any three terms of a proportion are given, the fourth may be found. Thus if $a$, $c$, $d$ are given, then $b=\frac{ad}{c}$.

Conversely, if there are any four quantities, $a$, $b$, $c$, $d$, such that $ad=bc$, then $a$, $b$, $c$, $d$ are proportionals; $a$ and $d$ being the extremes, $b$ and $c$ the means; or vice versâ.

297. DEFINITION. Quantities are said to be in **continued proportion** when the first is to the second, as the second is to the third, as the third to the fourth; and so on. Thus $a$, $b$, $c$, $d$, ...... are in continued proportion when

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\dots\dots\dots$$

If three quantities $a$, $b$, $c$ are in continued proportion, then

$$a : b=b : c;$$

$$\therefore \quad ac=b^2. \qquad \text{[Art. 296.]}$$

In this case $b$ is said to be a **mean proportional** between $a$ and $c$; and $c$ is said to be a **third proportional** to $a$ and $b$.

298. *If three quantities are proportionals the first is to the third in the duplicate ratio of the first to the second.*

Let the three quantities be $a$, $b$, $c$; then $\frac{a}{b}=\frac{b}{c}$.

Now $\frac{a}{c}=\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}$;

that is, $a : c=a^2 : b^2$.

299. *If* a : b = c : d *and* e : f = g : h *then will*

$$ae : bf=cg : dh.$$

For $\frac{a}{b}=\frac{c}{d}$ and $\frac{e}{f}=\frac{g}{h}$,

$$\therefore \quad \frac{ae}{bf}=\frac{cg}{dh}, \text{ or } ae : bf=cg : dh.$$

COR. If $a : b = c : d,$

and $b : x = d : y,$

then $a : x = c : y.$

This is the theorem known as *ex æquali* in Geometry.

300. If four quantities, $a$, $b$, $c$, $d$ form a proportion, many other proportions may be deduced by the properties of fractions. The results of these operations are very useful, and some of them are often quoted by the annexed names borrowed from Geometry.

(1) If $a : b = c : d$, then $b : a = d : c$. [*Invertendo.*]

For $\frac{a}{b} = \frac{c}{d}$; therefore $1 \div \frac{a}{b} = 1 \div \frac{c}{d}$;

that is $$\frac{b}{a} = \frac{d}{c};$$

or $$b : a = d : c.$$

(2) If $a : b = c : d$, then $a : c = b : d$. [*Alternando.*]

For $ad = bc$; therefore $\frac{ad}{cd} = \frac{bc}{cd}$;

that is, $$\frac{a}{c} = \frac{b}{d};$$

or $$a : c = b : d.$$

(3) If $a : b = c : d$, then $a + b : b = c + d : d$. [*Componendo.*]

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} + 1 = \frac{c}{d} + 1$;

that is, $$\frac{a+b}{b} = \frac{c+d}{d};$$

or $$a + b : b = c + d : d.$$

(4) If $a : b = c : d$, then $a - b : b = c - d : d$. [*Dividendo.*]

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} - 1 = \frac{c}{d} - 1$;

that is, $$\frac{a-b}{b} = \frac{c-d}{d};$$

or $$a - b : b = c - d : d.$$

(5) If $a : b = c : d$, then $a+b : a-b = c+d : c-d$.

For by (3) $$\frac{a+b}{b} = \frac{c+d}{d};$$

and by (4) $$\frac{a-b}{b} = \frac{c-d}{d};$$

$\therefore$ by division, $$\frac{a+b}{a-b} = \frac{c+d}{c-d};$$

or $$a+b : a-b = c+d : c-d.$$

Several other proportions may be proved in a similar way.

301. The results of the preceding article are the algebraical equivalents of some of the propositions in the fifth book of Euclid, and the student is advised to make himself familiar with them in their verbal form. For example, *dividendo* may be quoted as follows:

*When there are four proportionals, the excess of the first above the second is to the second, as the excess of the third above the fourth is to the fourth.*

302. We shall now compare the algebraical definition of proportion with that given in Euclid.

In algebraical symbols the definition of Euclid may be stated as follows:

Four magnitudes, $a$, $b$, $c$, $d$ are in proportion when $pa \gtreqless qb$ according as $pc \gtreqless qd$, $p$ and $q$ being *any positive integers whatever.*

I. To deduce the geometrical definition of proportion from the algebraical definition.

Since $\frac{a}{b} = \frac{c}{d}$, by multiplying both sides by $\frac{p}{q}$, we obtain

$$\frac{pa}{qb} = \frac{pc}{qd};$$

hence, from the properties of fractions,

$$pa \gtreqless qb \text{ according as } pc \gtreqless qd,$$

which proves the proposition.

II. To deduce the algebraical definition of proportion from the geometrical definition.

Given that $pa \gtreqless qb$ according as $pc \gtreqless qd$ to prove

$$\frac{a}{b} = \frac{c}{d}.$$

If $\frac{a}{b}$ is not equal to $\frac{c}{d}$, one of them must be the greater. Suppose $\frac{a}{b} > \frac{c}{d}$; then it will be possible to find some fraction $\frac{q}{p}$ which lies between them.

Hence $$\frac{a}{b} > \frac{q}{p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and $$\frac{c}{d} < \frac{q}{p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) $$pa > qb;$$
from (2) $$pc < qd;$$
and these contradict the hypothesis.

Therefore $\frac{a}{b}$ and $\frac{c}{d}$ are not unequal; that is $\frac{a}{b} = \frac{c}{d}$; which proves the proposition.

NOTE. The student should observe that the geometrical definition of proportion deals with *concrete* magnitudes, such as lines or areas, represented geometrically but not referred to any common unit. So that Euclid's definition is applicable to incommensurable as well as to commensurable quantities; whereas the algebraical definition, strictly speaking, applies only to commensurable quantities, since it tacitly assumes that $a$ is the same determinate multiple, part, or parts, of $b$ that $c$ is of $d$. But the proofs which have been given for commensurable quantities will still be true for incommensurables, since the ratio of two incommensurables can always be made to differ from the ratio of two integers *by less than any assignable quantity*. [See Art. 289.]

*Example* 1. If $$a : b = c : d = e : f,$$
shew that $$2a^2 + 3c^2 - 5e^2 : 2b^2 + 3d^2 - 5f^2 = ae : bf.$$

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$; then $a = bk$, $c = dk$, $e = fk$;

$$\therefore \frac{2a^2 + 3c^2 - 5e^2}{2b^2 + 3d^2 - 5f^2} = \frac{2b^2k^2 + 3d^2k^2 - 5f^2k^2}{2b^2 + 3d^2 - 5f^2}$$

$$= k^2 = \frac{a}{b} \times \frac{e}{f} = \frac{ae}{bf},$$

or $$2a^2 + 3c^2 - 5e^2 : 2b^2 + 3d^2 - 5f^2 = ae : bf.$$

*Example* 2. If

$$(3a+6b+c+2d)(3a-6b-c+2d)=(3a-6b+c-2d)(3a+6b-c-2d),$$

prove that $a$, $b$, $c$, $d$ are in proportion.

We have $$\frac{3a+6b+c+2d}{3a-6b+c-2d}=\frac{3a+6b-c-2d}{3a-6b-c+2d}.$$ [Art. 296.]

Componendo and Dividendo,

$$\frac{2(3a+c)}{2(6b+2d)}=\frac{2(3a-c)}{2(6b-2d)}.$$

Alternando, $$\frac{3a+c}{3a-c}=\frac{6b+2d}{6b-2d}.$$

Again, Componendo and Dividendo,

$$\frac{6a}{2c}=\frac{12b}{4d};$$

whence $$a:b=c:d.$$

*Example* 3. Solve the equation $\dfrac{x^2+x-2}{x-2}=\dfrac{4x^2+5x-6}{5x-6}$.

Dividendo, $$\frac{x^2}{x-2}=\frac{4x^2}{5x-6};$$

whence, dividing by $x^2$, which gives a solution $x=0$, [Art. 201.]

$$\frac{1}{x-2}=\frac{4}{5x-6};$$

whence, $$x=-2.$$

and therefore the roots are 0, $-2$.

## EXAMPLES XXXII. b.

Find a fourth proportional to

1. $a$, $ab$, $c$. 2. $a^2$, $2ab$, $3b^2$. 3. $x^3$, $xy$, $5x^2y$.

Find a third proportional to

4. $a^2b$, $ab$. 5. $x^3$, $2x^2$. 6. $3x$, $6xy$. 7. $1$, $x$.

Find a mean proportional between

8. $a^2$, $b^2$. 9. $2x^3$, $8x$. 10. $12ax^2$, $3a^3$. 11. $27a^2b^3$, $3b$.

If $a$, $b$, $c$ be three proportionals, shew that

12. $a : a+b=a-b : a-c$.

13. $(b^2+bc+c^2)(ac-bc+c^2)=b^4+ac^3+c^4$.

If $a : b = c : d$, prove that

14. $ab+cd : ab-cd = a^2+c^2 : a^2-c^2$.

15. $a^2+ac+c^2 : a^2-ac+c^2 = b^2+bd+d^2 : b^2-bd+d^2$.

16. $a : b = \sqrt{3a^2+5c^2} : \sqrt{3b^2+5d^2}$.

17. $\dfrac{a}{p}+\dfrac{b}{q} : a = \dfrac{c}{p}+\dfrac{d}{q} : c$.

18. $\dfrac{b}{a}+\dfrac{a}{b} : \dfrac{ab}{a^2+b^2} = \dfrac{d}{c}+\dfrac{c}{d} : \dfrac{cd}{c^2+d^2}$.

Solve the equations :

19. $3x-1 : 6x-7 = 7x-10 : 9x+10$.

20. $x-12 : y+3 = 2x-19 : 5y-13 = 5 : 14$.

21. $\dfrac{x^2-2x+3}{2x-3} = \dfrac{x^2-3x+5}{3x-5}$. 22. $\dfrac{2x-1}{x^2+2x-1} = \dfrac{x+4}{x^2+x+4}$.

23. If
$$(a+b-3c-3d)(2a-2b-c+d) = (2a+2b-c-d)(a-b-3c+3d)$$
prove that $a$, $b$, $c$, $d$ are proportionals.

24. If $a$, $b$, $c$, $d$ are in continued proportion, prove that
$$a : d = a^3+b^3+c^3 : b^3+c^3+d^3.$$

25. If $b$ is a mean proportional between $a$ and $c$, shew that $4a^2-9b^2$ is to $4b^2-9c^2$ in the duplicate ratio of $a$ to $b$.

26. If $a$, $b$, $c$, $d$ are in continued proportion, prove that $b+c$ is a mean proportional between $a+b$ and $c+d$.

27. If
$$a+b : b+c = c+d : d+a,$$
prove that $a=c$, or $a+b+c+d=0$.

## VARIATION.

303. DEFINITION. One quantity $A$ is said to **vary directly** as another $B$, when the two quantities depend upon each other in such a manner that if $B$ is changed, $A$ is changed *in the same proportion.*

NOTE. The word *directly* is often omitted, and $A$ is said to vary as $B$.

304. For instance: if a train moving at a uniform rate travels 40 miles in 60 minutes, it will travel 20 miles in 30 minutes, 80 miles in 120 minutes, and so on; the distance in each case being increased or diminished in the same ratio as the time. This is expressed by saying that when the velocity is uniform *the distance is proportional to the time*, or more briefly, *the distance varies as the time.*

Again, if we refer to the general formula of Art. 84, we find that $\frac{s}{t}=v$ is a relation connecting the space described by a body which moves for a time $t$ with *uniform* velocity $v$. That is, if $s_1, s_2, s_3 \ldots$ be spaces described in times $t_1, t_2, t_3 \ldots$ respectively,

we have $$\frac{s_1}{t_1}=\frac{s_2}{t_2}=\frac{s_3}{t_3}=\ldots\ldots=v.$$

From this it appears that the ratio of any value of $s$ to the corresponding value of $t$ is *constant*, that is, remains the same whatever numerical values $s$ and $t$ may have.

This is an instance of *direct variation*, and $s$ is said to *vary* as $t$.

305. The symbol $\propto$ is used to denote variation; so that $A \propto B$ is read "$A$ varies as $B$."

306. *If* A *varies as* B, *then* A *is equal to* B *multiplied by some constant quantity.*

For suppose that $a_1, a_2, a_3 \ldots$, $b_1, b_2, b_3 \ldots$ are corresponding values of $A$ and $B$.

Then, by definition, $\frac{A}{a_1}=\frac{B}{b_1}$; $\frac{A}{a_2}=\frac{B}{b_2}$; $\frac{A}{a_3}=\frac{B}{b_3}$; and so on,

$$\therefore \frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\ldots, \text{ each being equal to } \frac{A}{B}.$$

Hence $\dfrac{\text{any value of } A}{\text{the corresponding value of } B}$ is always the same;

that is, $$\frac{A}{B}=m, \text{ where } m \text{ is constant.}$$
$$\therefore A=mB.$$

307. DEFINITION. One quantity $A$ is said to **vary inversely** as another $B$ when $A$ varies *directly* as the reciprocal of $B$. [See Art. 107.]

Thus if $A$ varies inversely as $B$, $A=\frac{m}{B}$, where $m$ is constant.

The following is an illustration of inverse variation: If 6 men do a certain work in 8 hours, 12 men would do the same work in 4 hours, 2 men in 24 hours; and so on. Thus it appears that when the number of men is increased the time is proportionately decreased; and vice-versâ.

308. DEFINITION. One quantity is said to **vary jointly** as a number of others when it varies directly as their product.

Thus $A$ varies jointly as $B$ and $C$ when $A = mBC$. For instance, the interest on a sum of money varies jointly as the principal, the time, and the rate per cent.

309. DEFINITION. $A$ is said to vary directly as $B$ and inversely as $C$ when $A$ varies as $\frac{B}{C}$.

310. *If* A *varies as* B *when* C *is constant, and* A *varies as* C *when* B *is constant, then will* A *vary as* BC *when both* B *and* C *vary.*

The variation of $A$ depends partly on that of $B$ and partly on that of $C$. Suppose these latter variations to take place separately, each in its turn producing its own effect on $A$; also let $a$, $b$, $c$ be certain simultaneous values of $A$, $B$, $C$.

1. *Let $C$ be constant* while $B$ changes to $b$; then $A$ must undergo a partial change and will assume some intermediate value $a'$, where

$$\frac{A}{a'} = \frac{B}{b} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

2. *Let $B$ be constant*, that is, let it retain its value $b$, while $C$ changes to $c$; then $A$ must complete its change and pass from its intermediate value $a'$ to its final value $a$, where

$$\frac{a'}{a} = \frac{C}{c} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) and (2) $\quad \frac{A}{a'} \times \frac{a'}{a} = \frac{B}{b} \times \frac{C}{c}$;

that is, $\quad A = \frac{a}{bc} \cdot BC$,

or $\quad A$ varies as $BC$.

311. The following are illustrations of the theorem proved in the last article.

The amount of work done by *a given number of men* varies directly as the number of days they work, and the amount of work done *in a given time* varies directly as the number of men; therefore when the number of days and the number of men are both variable, the amount of work will vary as the product of the number of men and the number of days.

Again, in Geometry the area of a triangle varies directly as its base when the height is constant, and directly as the height when the base is constant; and when both the height and base are variable, the area varies as the product of the numbers representing the height and the base.

*Example* 1. If $A \propto B$, and $C \propto D$, then will $AC \propto BD$.

For, by supposition, $A=mB$, $C=nD$, where $m$ and $n$ are constants.

Therefore $AC=mnBD$; and as $mn$ is constant, $AC \propto BD$.

*Example* 2. If $x$ varies inversely as $y^2-1$, and is equal to 24 when $y=10$; find $x$ when $y=5$.

By supposition, $x=\dfrac{m}{y^2-1}$, where $m$ is constant.

Putting $x=24$, $y=10$, we obtain $24=\dfrac{m}{99}$,

whence $$m=24 \times 99.$$

$$\therefore\ x=\frac{24 \times 99}{y^2-1};$$

hence, putting $y=5$, we obtain $x=99$.

*Example* 3. The volume of a pyramid varies jointly as its height and the area of its base; and when the area of the base is 60 square feet and the height 14 feet the volume is 280 cubic feet. What is the area of the base of a pyramid whose volume is 390 cubic feet and whose height is 26 feet?

Let $V$ denote the volume, $A$ the area of the base, and $h$ the height;

then $V=mAh$, where $m$ is constant.

Substituting the given values of $V$, $A$, $h$ we have

$$280=m \times 60 \times 14;$$

$$\therefore\ m=\frac{280}{60 \times 14}=\frac{1}{3}.$$

$$\therefore\ V=\frac{1}{3}Ah.$$

Also when $$V=390,\ h=26;$$

$$\therefore\ 390=\frac{1}{3}A \times 26;$$

$$\therefore\ A=45.$$

Hence the area of the base is 45 square feet.

## EXAMPLES XXXII. c.

1. If $x \propto y$, and $y=7$ when $x=18$, find $x$ when $y=21$.

2. If $x \propto y$, and $y=3$ when $x=2$, find $y$ when $x=18$.

3. $A$ varies jointly as $B$ and $C$; and $A=6$ when $B=3$, $C=2$: find $A$ when $B=5$, $C=7$.

4. $A$ varies jointly as $B$ and $C$; and $A=9$ when $B=5$, $C=7$: find $B$ when $A=54$, $C=10$.

5. If $x \propto \frac{1}{y}$, and $y=4$ when $x=15$, find $y$ when $x=6$.

6. If $y \propto \frac{1}{x}$, and $y=1$ when $x=1$, find $x$ when $y=5$.

7. $A$ varies as $B$ directly, and as $C$ inversely; and $A=10$ when $B=15$, $C=6$; find $A$ when $B=8$, $C=2$.

8. If $x$ varies as $y$ directly, and as $z$ inversely, and $x=14$ when $y=10$, $z=14$; find $z$ when $x=49$, $y=45$.

9. If $x \propto \frac{1}{y}$, and $y \propto \frac{1}{z}$, prove that $z \propto x$.

10. If $a \propto b$, prove that $a^n \propto b^n$.

11. If $x \propto z$ and $y \propto z$, prove that $x^2-y^2 \propto z^2$.

12. If $3a+7b \propto 3a+13b$, and when $a=5$, $b=3$, find the equation between $a$ and $b$.

13. If $5x-y \propto 10x-11y$, and when $x=7$, $y=5$, find the equation between $x$ and $y$.

14. If the cube of $x$ varies as the square of $y$, and if $x=3$ when $y=5$, find the equation between $x$ and $y$.

15. If the square root of $a$ varies as the cube root of $b$, and if $a=4$ when $b=8$, find the equation between $a$ and $b$.

16. If $y$ varies inversely as the square of $x$, and if $y=8$ when $x=3$, find $x$ when $y=2$.

17. If $x \propto y+a$, where $a$ is constant, and $x=15$ when $y=1$, and $x=35$ when $y=5$; find $x$ when $y=2$.

18. If $a+b \propto a-b$, prove that $a^2+b^2 \propto ab$; and if $a \propto b$, prove that $a^2-b^2 \propto ab$.

**19.** If $y$ be the sum of three quantities which vary as $x$, $x^2$, $x^3$ respectively, and when $x=1$, $y=4$, when $x=2$, $y=8$, and when $x=3$, $y=18$, express $y$ in terms of $x$.

**20.** Given that the area of a circle varies as the square of its radius, and that the area of a circle is 154 square feet when the radius is 7 feet; find the area of a circle whose radius is 10 feet 6 inches.

**21.** The area of a circle varies as the square of its diameter; prove that the area of a circle whose diameter is $2\frac{1}{2}$ inches is equal to the sum of the areas of two circles whose diameters are $1\frac{1}{2}$, and 2 inches respectively.

**22.** The pressure of wind on a plane surface varies jointly as the area of the surface, and the square of the wind's velocity. The pressure on a square foot is 1 lb. when the wind is moving at the rate of 15 miles per hour; find the velocity of the wind when the pressure on a square yard is 16 lbs.

**23.** The value of a silver coin varies directly as the square of its diameter, while its thickness remains the same; it also varies directly as its thickness while its diameter remains the same. Two silver coins have their diameters in the ratio of 4 : 3. Find the ratio of their thicknesses if the value of the first be four times that of the second.

**24.** The volume of a circular cylinder varies as the square of the radius of the base when the height is the same, and as the height when the base is the same. The volume is 88 cubic feet when the height is 7 feet, and the radius of the base is 2 feet; what will be the height of a cylinder on a base of radius 9 feet, when the volume is 396 cubic feet?

# CHAPTER XXXIII.

## ARITHMETICAL PROGRESSION.

312. DEFINITION. Quantities are said to be in **Arithmetical Progression** when they increase or decrease by a *common difference.*

Thus each of the following series forms an Arithmetical Progression:

$$3,\ 7,\ 11,\ 15, \ldots\ldots\ldots\ldots$$
$$8,\ 2,\ -4,\ -10,\ \ldots\ldots\ldots\ldots$$
$$a,\ a+d,\ a+2d,\ a+3d, \ldots\ldots$$

The common difference is found by subtracting *any* term of the series from that which *follows* it. In the first of the above examples the common difference is 4; in the second it is $-6$; in the third it is $d$.

313. If we examine the series

$$a,\ a+d,\ a+2d,\ a+3d, \ldots$$

we notice that *in any term the coefficient of* d *is always less by one than the number of the term in the series.*

Thus the 3rd term is $a+2d$;
6th term is $a+5d$;
20th term is $a+19d$;
and, generally, the $p$th term is $a+(p-1)d$.

If $n$ be the number of terms, and if $l$ denote the last, or $n$th term, we have $l=a+(n-1)d$.

314. *To find the sum of a number of terms in Arithmetical Progression.*

Let $a$ denote the first term, $d$ the common difference, and $n$ the number of terms. Also let $l$ denote the last term, and $s$ the required sum; then

$$s=a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l;$$

and, by writing the series in the reverse order,

$$s=l+(l-d)+(l-2d)+\ldots+(a+2d)+(a-d)+a.$$

Adding together these two series,

$$2s=(a+l)+(a+l)+(a+l)+\dots \text{ to } n \text{ terms}$$
$$=n(a+l),$$
$$\therefore\ s=\frac{n}{2}(a+l)\ \dots\dots\dots\dots(1);$$

and
$$l=a+(n-1)d\ \dots\dots\dots\dots(2).$$
$$\therefore\ s=\frac{n}{2}\{2a+(n-1)d\}\ \dots\dots\dots\dots(3).$$

315. In the last article we have three useful formulæ (1), (2), (3); in each of these any one of the letters may denote the unknown quantity when the three others are known. [See Art. 82, Chap. IX.] For instance, in (1) if we substitute given values for $s$, $n$, $l$, we obtain an equation for finding $a$; and similarly in the other formulæ. But it is necessary to guard against a too mechanical use of these general formulæ, and it will often be found better to solve simple questions by a mental rather than by an actual reference to the requisite formula.

*Example* 1. Find the 20th and 35th terms of the series

38, 36, 34,......

Here the common-difference is 36 – 38, or – 2.

$$\therefore \text{ the } 20^{\text{th}} \text{ term}=38+19(-2)$$
$$=0;$$
$$\text{and the } 35^{\text{th}} \text{ term}=38+34(-2)$$
$$=-30.$$

*Example* 2. Find the sum of the series $5\frac{1}{2}$, $6\frac{3}{4}$, 8,...... to 17 terms.

Here the common difference is $1\frac{1}{4}$; hence from (3)

The sum
$$=\frac{17}{2}\left\{2\times\frac{11}{2}+16\times1\tfrac{1}{4}\right\}$$
$$=\frac{17}{2}(11+20)$$
$$=\frac{17\times31}{2}$$
$$=263\tfrac{1}{2}.$$

*Example* 3. The first term of a series is 5, the last 45, and the sum 400: find the number of terms, and the common difference.

If $n$ be the number of terms, then from (2)

$$400=\frac{n}{2}(5+45);$$

whence $$n=16.$$

If $d$ be the common difference

$$\begin{aligned}45 &= \text{the } 16^{\text{th}} \text{ term}\\ &=5+15d;\end{aligned}$$

whence $$d=2\tfrac{2}{3}.$$

## EXAMPLES XXXIII. a.

1. Find the 27th and 41st terms in the series 5, 11, 17,...
2. Find the 13th and 109th terms in the series 71, 70, 69,...
3. Find the 17th and 54th terms in the series 10, $11\frac{1}{2}$, 13,...
4. Find the 20th and 13th terms in the series $-3$, $-2$, $-1$,...
5. Find the 90th and 16th terms in the series $-4$, 2·5, 9,...
6. Find the 37th and 89th terms in the series $-2·8$, 0, 2·8,...

Find the last term in the following series:

**7.** 5, 7, 9,... to 20 terms. **8.** 7, 3, $-1$,... to 15 terms.

**9.** $13\frac{1}{2}$, 9, $4\frac{1}{2}$,... to 13 terms. **10.** ·6, 1·2, 1·8,... to 12 terms.

**11.** 2·7, 3·4, 4·1,... to 11 terms. **12.** $x$, $2x$, $3x$,... to 25 terms.

**13.** $a-d$, $a+d$, $a+3d$,... to 30 terms.

**14.** $2a-b$, $4a-3b$, $6a-5b$,... to 40 terms.

Find the last term and sum of the following series:

**15.** 14, 64, 114,... to 20 terms. **16.** 1, 1·2, 1·4,... to 12 terms.

**17.** 9, 5, 1,... to 100 terms. **18.** $\frac{1}{4}$, $-\frac{1}{4}$, $-\frac{3}{4}$,... to 21 terms.

**19.** $3\frac{1}{2}$, 1, $-1\frac{1}{2}$,... to 19 terms. **20.** 64, 96, 128,... to 16 terms.

Find the sum of the following series:

**21.** 5, 9, 13,... to 19 terms. **22.** 12, 9, 6,... to 23 terms.

**23.** 4, $5\frac{1}{4}$, $6\frac{1}{2}$,... to 37 terms. **24.** $10\frac{1}{2}$, 9, $7\frac{1}{2}$,... to 94 terms.

**25.** $-3$, 1, 5,... to 17 terms. **26.** 10, $9\frac{2}{3}$, $9\frac{1}{3}$,... to 21 terms.

**27.** $p$, $3p$, $5p$,... to $p$ terms. **28.** $3a$, $a$, $-a$,... to $a$ terms.

**29.** $a$, 0, $-a$,... to $a$ terms. **30.** $-3q$, $-q$, $q$,... to $p$ terms.

Find the number of terms and the common difference when

**31.** The first term is 3, the last term 90, and the sum 1395.

**32.** The first term is 79, the last term 7, and the sum 1075.

**33.** The sum is 24, the first term 9, the last term $-6$.

**34.** The sum is 714, the first term 1, the last term $58\frac{1}{2}$.

**35.** The last term is $-16$, the sum $-133$, the first term $-3$.

**36.** The first term is $-75$, the sum $-740$, the last term 1.

**37.** The first term is $a$, the last $13a$, and the sum $49a$.

**38.** The sum is $-320x$, the first term $3x$, the last term $-35x$.

316. If *any two* terms of an Arithmetical Progression be given, the series can be completely determined; for the data furnish *two* simultaneous equations, the solution of which will give the first term and the common difference.

*Example.* Find the series whose 7th and 51st terms are $-3$ and $-355$ respectively.

If $a$ be the first term, and $d$ the common difference,

$$-3 = \text{the } 7^{\text{th}} \text{ term}$$
$$= a + 6d;$$

and
$$-355 = \text{the } 51^{\text{st}} \text{ term}$$
$$= a + 50d;$$

whence, by subtraction, $-352 = 44d$;

$\therefore d = -8$; and consequently $a = 45$.

Hence the series is 45, 37, 29......

317. Definition. When three quantities are in Arithmetical Progression the middle one is said to be the **Arithmetic mean** of the other two.

Thus $a$ is the arithmetic mean between $a-d$ and $a+d$.

318. *To find the arithmetic mean between two given quantities.*

Let $a$ and $b$ be the two quantities; $A$ the arithmetic mean. Then since $a$, $A$, $b$ are in A.P. we must have

$$b - A = A - a$$

each being equal to the common difference;

whence
$$A = \frac{a+b}{2}.$$

319. Between two given quantities it is always possible to insert any number of terms such that the whole series thus formed shall be in A.P.; and by an extension of the definition in Art. 317, the terms thus inserted are called the *arithmetic means.*

*Example.* Insert 20 arithmetic means between 4 and 67.

Including the extremes the number of terms will be 22; so that we have to find a series of 22 terms in A.P., of which 4 is the first and 67 the last.

Let $d$ be the common difference;

then $$67 = \text{the } 22^{\text{nd}} \text{ term}$$
$$= 4 + 21d;$$

whence $d=3$, and the series is 4, 7, 10, ......61, 64, 67;

and the required means are 7, 10, 13, ......58, 61, 64.

320. *To insert a given number of arithmetic means between two given quantities.*

Let $a$ and $b$ be the given quantities, $n$ the number of means.

Including the extremes the number of terms will be $n+2$; so that we have to find a series of $n+2$ terms in A.P., of which $a$ is the first, and $b$ is the last.

Let $d$ be the common difference;

then $$b = \text{the } (n+2)^{\text{th}} \text{ term}$$
$$= a + (n+1)d;$$

whence $$d = \frac{b-a}{n+1};$$

and the required means are

$$a + \frac{b-a}{n+1}, \quad a + \frac{2(b-a)}{n+1}, \ldots\ldots a + \frac{n(b-a)}{n+1}.$$

*Example* 1. Find the 30th term of an A.P. of which the first term is 17, and the 100th term $-16$.

Let $d$ be the common difference;

then $$-16 = \text{the } 100^{\text{th}} \text{ term}$$
$$= 17 + 99d;$$
$$\therefore\ d = -\frac{1}{3}.$$

The 30th term $$= 17 + 29\left(-\frac{1}{3}\right)$$
$$= 7\tfrac{1}{3}.$$

*Example* 2. The sum of three numbers in A.P. is 33, and their product is 792; find them.

Let $a$ be the *middle* number, $d$ the common difference; then the three numbers are $a-d$, $a$, $a+d$.

Hence $$a-d+a+a+d=33;$$
whence $a=11$; and the three numbers are $11-d$, 11, $11+d$.

$$\therefore\ 11(11+d)(11-d)=792,$$
$$121-d^2=72,$$
$$d=\pm 7;$$
and the numbers are 4, 11, 18.

*Example* 3. How many terms of the series 24, 20, 16,...... must be taken that the sum may be 72?

Let the number of terms be $n$; then, since the common difference is $20-24$, or $-4$, we have from (3), Art. 314,

$$72=\frac{n}{2}\{2\times 24+(n-1)(-4)\}$$
$$=24n-2n(n-1);$$
whence $$n^2-13n+36=0,$$
or $$(n-4)(n-9)=0;$$
$$\therefore\ n=4 \text{ or } 9.$$

Both these values satisfy the conditions of the question; for if we write down the first 9 terms, we get 24, 20, 16, 12, 8, 4, 0, $-4$, $-8$; and, as the last five terms destroy each other, the sum of 9 terms is the same as that of 4 terms.

*Example* 4. An A.P. consists of 21 terms; the sum of the three terms in the middle is 129, and of the last three is 237; find the series.

Let $a$ be the first term, and $d$ the common difference. Then

$$237=\text{the sum of the last three terms}$$
$$=a+20d+a+19d+a+18d$$
$$=3a+57d;$$
whence $$a+19d=79 \text{ .............................}(1).$$

Again, the three middle terms are the $10^{\text{th}}$, $11^{\text{th}}$, $12^{\text{th}}$;

hence $$129=\text{the sum of the three middle terms}$$
$$=a+9d+a+10d+a+11d$$
$$=3a+30d;$$
whence $$a+10d=43 \text{ ............................}(2).$$

From (1) and (2), we obtain $d=4$, $a=3$.

Hence the series is 3, 7, 11,......83.

## EXAMPLES XXXIII. b.

Find the series in which

1. The 27th term is 186, and the 45th term 312.
2. The 5th term is 1, and the 31st term $-77$.
3. The 15th term is $-25$, and the 23rd term $-41$.
4. The 9th term is $-11$, and the 102nd term $-150\frac{1}{2}$.
5. The 15th term is 25, and the 29th term 46.
6. The 16th term is 214, and the 51st term 739.
7. The 3rd and 7th terms of an A.P. are 7 and 19; find the 15th term.
8. The 54th and 4th terms are $-125$ and 0; find the 42nd term.
9. The 31st and 2nd terms are $\frac{1}{2}$ and $7\frac{3}{4}$; find the 59th term.
10. Insert 15 arithmetic means between 71 and 23.
11. Insert 17 arithmetic means between 93 and 69.
12. Insert 14 arithmetic means between $-7\frac{1}{5}$ and $-2\frac{1}{5}$.
13. Insert 16 arithmetic means between 7·2 and $-6{\cdot}4$.
14. Insert 36 arithmetic means between $8\frac{1}{2}$ and $2\frac{1}{3}$.

How many terms must be taken of

15. The series 42, 39, 36,… to make 315?
16. The series $-16$, $-15$, $-14$,… to make $-100$?
17. The series $15\frac{2}{3}$, $15\frac{1}{3}$, 15,… to make 129?
18. The series 20, $18\frac{3}{4}$, $17\frac{1}{2}$,… to make $162\frac{1}{2}$?
19. The series $-10\frac{1}{2}$, $-9$, $-7\frac{1}{2}$,… to make $-42$?
20. The series $-6\frac{4}{5}$, $-6\frac{2}{5}$, $-6$,… to make $-52\frac{4}{5}$?
21. The sum of three numbers in A.P. is 39, and their product is 2184; find them.
22. The sum of three numbers in A.P. is 12, and the sum of their squares is 66; find them.
23. The sum of five numbers in A.P. is 75, and the product of the greatest and least is 161; find them.
24. The sum of five numbers in A.P. is 40, and the sum of their squares is 410; find them.
25. The 12th, 85th and last terms of an A.P. are 38, 257, 395 respectively; find the number of terms.

# CHAPTER XXXIV.

## GEOMETRICAL PROGRESSION.

321. DEFINITION. Quantities are said to be in **Geometrical Progression** when they increase or decrease by a *constant factor*.

Thus each of the following series forms a Geometrical Progression:

$$3,\ 6,\ 12,\ 24,\ \ldots\ldots\ldots\ldots\ldots\ldots$$

$$1,\quad -\frac{1}{3},\quad \frac{1}{9},\quad -\frac{1}{27},\ldots\ldots\ldots$$

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots\ldots\ldots\ldots\ldots$$

The constant factor is also called the *common ratio*, and it is found by dividing *any* term by that which immediately *precedes* it. In the first of the above examples the common ratio is 2; in the second it is $-\frac{1}{3}$; in the third it is $r$.

322. If we examine the series

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ldots\ldots$$

we notice that *in any term the index of* r *is always less by one than the number of the term in the series.*

Thus the 3rd term is $ar^2$;

the 6th term is $ar^5$;

the 20th term is $ar^{19}$;

and, generally, the $p$th term is $ar^{p-1}$.

If $n$ be the number of terms, and if $l$ denote the last, or $n$th term, we have $l = ar^{n-1}$.

*Example.* Find the 8th term of the series $-\frac{1}{3},\ \frac{1}{2},\ -\frac{3}{4},\ldots$

The common ratio is $\frac{1}{2} \div \left(-\frac{1}{3}\right)$, or $-\frac{3}{2}$;

$$\therefore \text{ the 8th term} = -\frac{1}{3} \times \left(-\frac{3}{2}\right)^7$$

$$= -\frac{1}{3} \times -\frac{2187}{128}$$

$$= \frac{729}{128}.$$

323. DEFINITION. When three quantities are in Geometrical Progression the middle one is called the **geometric mean** between the other two.

*To find the geometric mean between two given quantities.*

Let $a$ and $b$ be the two quantities; $G$ the geometric mean. Then since $a$, $G$, $b$ are in G.P.,

$$\frac{b}{G}=\frac{G}{a},$$

each being equal to the common ratio;

$$\therefore\ G^2=ab;$$

whence $$G=\sqrt{ab}.$$

324. *To insert a given number of geometric means between two given quantities.*

Let $a$ and $b$ be the given quantities, $n$ the number of means.

In all there will be $n+2$ terms; so that we have to find a series of $n+2$ terms in G.P., of which $a$ is the first and $b$ the last.

Let $r$ be the common ratio;

then $$b=\text{the } (n+2)^{\text{th}} \text{ term}$$
$$=ar^{n+1};$$
$$\therefore\ r^{n+1}=\frac{b}{a};$$
$$\therefore\ r=\left(\frac{b}{a}\right)^{\frac{1}{n+1}} \quad \dots\dots\dots\dots(1).$$

Hence the required means are $ar$, $ar^2$, $\dots ar^n$, where $r$ has the value found in (1).

*Example.* Insert 4 geometric means between 160 and 5.

We have to find 6 terms in G. P. of which 160 is the first, and 5 the sixth.

Let $r$ be the common ratio;

$$\text{then } 5=\text{the sixth term}$$
$$=160r^5;$$
$$\therefore\ r^5=\frac{1}{32};$$

whence, *by trial,* $$r=\frac{1}{2};$$

and the means are 80, 40, 20, 10.

325. *To find the sum of a number of terms in Geometrical Progression.*

Let $a$ be the first term, $r$ the common ratio, $n$ the number of terms, and $s$ the sum required. Then

$$s = a + ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1};$$

multiplying every term by $r$, we have

$$rs = ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n.$$

Hence by subtraction,

$$rs - s = ar^n - a;$$

$$\therefore\ (r-1)s = a(r^n - 1);$$

$$\therefore\ s = \frac{a(r^n-1)}{r-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Changing the signs in numerator and denominator [Art. 170.]

$$s = \frac{a(1-r^n)}{1-r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

NOTE. It will be found convenient to remember both forms given above for $s$, using (2) in all cases except when $r$ is *positive and greater than* 1.

Since $ar^{n-1} = l$, the formula (1) may be written

$$s = \frac{rl - a}{r-1};$$

a form which is sometimes useful.

*Example* 1. Sum the series 81, 54, 36, ...... to 9 terms.

The common ratio $= \frac{54}{81} = \frac{2}{3}$, which is less than 1;

hence the sum
$$= \frac{81\left\{1 - \left(\frac{2}{3}\right)^9\right\}}{1 - \frac{2}{3}}$$

$$= 243\left\{1 - \left(\frac{2}{3}\right)^9\right\}$$

$$= 243 - \frac{512}{81}$$

$$= 236\frac{55}{81}.$$

*Example* 2. Sum the series $\frac{2}{3}$, $-1$, $\frac{3}{2}$, ...... to 7 terms.

The common ratio $= -\frac{3}{2}$; hence by formula (2)

$$\text{the sum} = \frac{\frac{2}{3}\left\{1-\left(-\frac{3}{2}\right)^7\right\}}{1+\frac{3}{2}}$$

$$= \frac{\frac{2}{3}\left\{1+\frac{2187}{128}\right\}}{\frac{5}{2}}$$

$$= \frac{2}{3} \times \frac{2315}{128} \times \frac{2}{5}$$

$$= \frac{463}{96}.$$

## EXAMPLES XXXIV. a.

1. Find the 5th and 8th terms of the series 3, 6, 12,...
2. Find the 10th and 16th terms of the series 256, 128, 64,...
3. Find the 7th and 11th terms of the series 64, $-32$, 16,...
4. Find the 8th and 12th terms of the series 81, $-27$, 9,...
5. Find the 14th and 7th terms of the series $\frac{1}{64}$, $\frac{1}{32}$, $\frac{1}{16}$,...
6. Find the 4th and 8th terms of the series ·008, ·04, ·2,...

Find the last term in the following series:

7. 2, 4, 8,... to 9 terms.
8. 2, $-6$, 18,... to 8 terms.
9. 2, 3, $4\frac{1}{2}$,... to 6 terms.
10. 3, $-3^2$, $3^3$,... to $2n$ terms.
11. $x$, $x^3$, $x^5$,... to $p$ terms.
12. $x$, 1, $\frac{1}{x}$,... to 30 terms.
13. Insert 3 geometric means between 486 and 6.
14. Insert 4 geometric means between $\frac{1}{8}$ and 128.
15. Insert 6 geometric means between 56 and $-\frac{7}{16}$.
16. Insert 5 geometric means between $\frac{32}{81}$ and $4\frac{1}{2}$.

Find the last term and the sum of the following series:

**17.** $3, 6, 12, \ldots$ to 8 terms. **18.** $6, -18, 54, \ldots$ to 6 terms.

**19.** $64, 32, 16, \ldots$ to 10 terms. **20.** $8{\cdot}1, 2{\cdot}7, {\cdot}9, \ldots$ to 7 terms.

**21.** $\frac{1}{72}, \frac{1}{24}, \frac{1}{8}, \ldots$ to 8 terms. **22.** $4\frac{1}{2}, 1\frac{1}{2}, \frac{1}{2}, \ldots$ to 9 terms.

Find the sum of the series

**23.** $3, -1, \frac{1}{3}, \ldots$ to 6 terms. **24.** $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \ldots$ to 7 terms.

**25.** $-\frac{2}{5}, \frac{1}{2}, -\frac{5}{8}, \ldots$ to 6 terms. **26.** $1, -\frac{1}{2}, \frac{1}{4}, \ldots$ to 12 terms.

**27.** $9, -6, 4, \ldots$ to 7 terms. **28.** $\frac{2}{3}, -\frac{1}{6}, \frac{1}{24}, \ldots$ to 8 terms.

**29.** $1, 3, 3^2, \ldots$ to $p$ terms. **30.** $2, -4, 8, \ldots$ to $2p$ terms.

**31.** $\frac{1}{\sqrt{3}}, 1, \frac{3}{\sqrt{3}}, \ldots$ to 8 terms. **32.** $\sqrt{a}, \sqrt{a^3}, \sqrt{a^5}, \ldots$ to $a$ terms.

**33.** $\frac{1}{\sqrt{2}}, -2, \frac{8}{\sqrt{2}}, \ldots$ to 7 terms. **34.** $\sqrt{2}, \sqrt{6}, 3\sqrt{2}, \ldots$ to 12 terms.

326. Consider the series $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\ldots$

The sum to $n$ terms

$$= \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}$$

$$= 2\left(1 - \frac{1}{2^n}\right)$$

$$= 2 - \frac{1}{2^{n-1}}.$$

From this result it appears that however many terms be taken the sum of the above series is always less than 2. Also we see that, by making $n$ sufficiently large, we can make the fraction $\frac{1}{2^{n-1}}$ as small as we please. Thus by taking a sufficient number of terms the sum can be made to differ by as little as we please from 2.

In the next article a more general case is discussed.

327. From Art. 325 we have $s = \dfrac{a(1-r^n)}{1-r}$

$$= \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Suppose $r$ is a proper fraction; then the greater the value of $n$ the smaller is the value of $r^n$, and consequently of $\dfrac{ar^n}{1-r}$; and therefore by making $n$ sufficiently large, we can make the sum of $n$ terms of the series differ from $\dfrac{a}{1-r}$ by as small a quantity as we please.

This result is usually stated thus: *the sum of an infinite number of terms of a decreasing Geometrical Progression is* $\dfrac{\mathrm{a}}{1-\mathrm{r}}$; or more briefly, *the sum to infinity is* $\dfrac{\mathrm{a}}{1-\mathrm{r}}$.

328. Recurring decimals furnish a good illustration of infinite Geometrical Progressions.

*Example.* Find the value of $\cdot 4\dot{2}\dot{3}$.

$$\cdot 4\dot{2}\dot{3} = \cdot 4232323\ldots\ldots$$

$$= \frac{4}{10} + \frac{23}{1000} + \frac{23}{100000} + \ldots\ldots$$

$$= \frac{4}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \ldots\ldots$$

$$= \frac{4}{10} + \frac{23}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\ldots\right)$$

$$= \frac{4}{10} + \frac{23}{10^3} \cdot \frac{1}{1 - \dfrac{1}{10^2}}$$

$$= \frac{4}{10} + \frac{23}{10^3} \cdot \frac{100}{99}$$

$$= \frac{4}{10} + \frac{23}{990}$$

$$= \frac{419}{990},$$

which agrees with the value found by the usual arithmetical rule.

## EXAMPLES XXXIV. b.

Sum to infinity the following series:

**1.** $9,\ 6,\ 4, \ldots$ **2.** $12,\ 6,\ 3, \ldots$

**3.** $\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8}, \ldots$ **4.** $\frac{1}{2},\ -\frac{1}{4},\ \frac{1}{8}, \ldots$

**5.** $\frac{1}{3},\ \frac{2}{9},\ \frac{4}{27}, \ldots$ **6.** $\frac{8}{5},\ -1,\ \frac{5}{8}, \ldots$

**7.** $\cdot 9,\ \cdot 03,\ \cdot 001, \ldots$ **8.** $\cdot 8,\ -\cdot 4,\ \cdot 2, \ldots$

Find by the method of Art. 328, the value of

**9.** $\cdot\dot{3}$. **10.** $\cdot 1\dot{6}$. **11.** $\cdot\dot{2}\dot{4}$. **12.** $\cdot 3\dot{7}\dot{8}$. **13.** $\cdot\dot{0}3\dot{7}$.

Find the series in which

**14.** The 10th term is 320 and the 6th term 20.

**15.** The 5th term is $\frac{27}{16}$ and the 9th term is $\frac{1}{3}$.

**16.** The 7th term is 625 and the 4th term $-5$.

**17.** The 3rd term is $\frac{9}{16}$ and the 6th term $-4\frac{1}{2}$.

**18.** Divide 183 into three parts in G.P. such that the sum of the first and third is $2\frac{1}{20}$ times the second.

**19.** Shew that the product of any odd number of consecutive terms of a G.P. will be equal to the $n^{\text{th}}$ power of the middle term, $n$ being the number of terms.

**20.** The first two terms of an infinite G.P. are together equal to 1, and every term is twice the sum of all the terms which follow. Find the series.

Sum the following series:

**21.** $y^2+2b,\ y^4+4b,\ y^6+6b, \ldots$ to $n$ terms.

**22.** $\frac{3+2\sqrt{2}}{3-2\sqrt{2}},\ 1,\ \frac{3-2\sqrt{2}}{3+2\sqrt{2}}, \ldots$ to infinity.

**23.** $\sqrt{\frac{3}{2}},\ \frac{1}{3}\sqrt{2},\ \frac{2}{9}\sqrt{\frac{2}{3}}, \ldots$ to infinity.

**24.** $2n-\frac{1}{2},\ 4n+\frac{1}{6},\ 6n-\frac{1}{18}, \ldots$ to $2n$ terms.

# CHAPTER XXXV.

## HARMONICAL PROGRESSION.

329. DEFINITION. Three quantities $a$, $b$, $c$ are said to be in **Harmonical Progression** when $\frac{a}{c}=\frac{a-b}{b-c}$.

Any number of quantities are said to be in Harmonical Progression when every three consecutive terms are in Harmonical Progression.

330. *The reciprocals of quantities in Harmonical Progression are in Arithmetical Progression.*

By definition, if $a$, $b$, $c$ are in Harmonical Progression,

$$\frac{a}{c}=\frac{a-b}{b-c};$$

$$\therefore\ a(b-c)=c(a-b),$$

dividing every term by $abc$,

$$\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a},$$

which proves the proposition.

331. Harmonical properties are chiefly interesting because of their importance in Geometry and in the Theory of Sound: in Algebra the proposition just proved is the only one of any importance. There is no general formula for the sum of any number of quantities in Harmonical Progression. Questions in H.P. are generally solved by inverting the terms, and making use of the properties of the corresponding A.P.

*Example.* The 12th term of a H.P. is $\frac{1}{5}$, and the 19th term is $\frac{3}{22}$: find the series.

Let $a$ be the first term, $d$ the common difference of the corresponding A.P.; then

$$5 = \text{the 12th term}$$
$$= a + 11d;$$

and

$$\frac{22}{3} = \text{the 19th term}$$
$$= a + 18d;$$

whence

$$d = \frac{1}{3}, \quad a = \frac{4}{3}.$$

Hence the Arithmetical Progression is $\frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \ldots\ldots;$

and the Harmonical Progression is $\frac{3}{4}, \frac{3}{5}, \frac{1}{2}, \frac{3}{7}, \ldots\ldots$

332. *To find the harmonic mean between two given quantities.*

Let $a$, $b$ be the two quantities, $H$ their harmonic mean; then $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P.,

$$\therefore \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H},$$
$$\frac{2}{H} = \frac{1}{a} + \frac{1}{b},$$
$$H = \frac{2ab}{a+b}.$$

333. If $A$, $G$, $H$ be the arithmetic, geometric, and harmonic means between $a$ and $b$, we have proved

$$A = \frac{a+b}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$
$$G = \sqrt{ab} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$
$$H = \frac{2ab}{a+b} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Therefore

$$AH = \frac{a+b}{2} \cdot \frac{2ab}{a+b}$$
$$= ab$$
$$= G^2;$$

that is, $G$ is the geometric mean between $A$ and $H$.

334. Miscellaneous questions in the Progressions afford scope for much skill and ingenuity, the solution being often very neatly effected by some special artifice. The student will find the following hints useful.

1. If the same quantity be added to, or subtracted from, all the terms of an A.P., the resulting terms will form an A.P. with the same common difference as before. [Art. 312.]

2. If all the terms of an A.P. be multiplied or divided by the same quantity, the resulting terms form an A.P., but with a new common difference. [Art. 312.]

3. If all the terms of a G.P. be multiplied or divided by the same quantity, the resulting terms form a G.P. with the same common ratio as before. [Art. 322.]

4. If $a, b, c, d \ldots$ be in G.P., they are also in *continued proportion*, since, by definition,

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \ldots\ldots = \frac{1}{r}.$$

We subjoin a few examples worked out to illustrate useful methods.

*Example* 1. Find three quantities in G.P. such that their product is 343, and their sum $30\frac{1}{3}$.

Let $\frac{a}{r}$, $a$, $ar$ be the three quantities;

then we have $$\frac{a}{r} \times a \times ar = 343 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and $$a\left(\frac{1}{r} + 1 + r\right) = \frac{91}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) $$a^3 = 343,$$
$$a = 7;$$

$\therefore$ from (2) $$7(1 + r + r^2) = \frac{91}{3} r.$$

Whence we obtain $$r = 3, \text{ or } \frac{1}{3}.$$

and the numbers are $\frac{7}{3}$, 7, 21.

*Example* 2. If $a$, $b$, $c$ be in H.P., prove that $\frac{a}{b+c}$, $\frac{b}{c+a}$, $\frac{c}{a+b}$ are also in H. P.

Since $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.,

$$\frac{a+b+c}{a}, \quad \frac{a+b+c}{b}, \quad \frac{a+b+c}{c} \text{ are in A.P.};$$

$$\therefore\ 1+\frac{b+c}{a}, \ 1+\frac{a+c}{b}, \ 1+\frac{a+b}{c} \text{ are in A.P.};$$

$$\therefore\ \frac{b+c}{a}, \quad \frac{a+c}{b}, \quad \frac{a+b}{c} \text{ are in A.P.};$$

$$\therefore\ \frac{a}{b+c}, \quad \frac{b}{c+a}, \quad \frac{c}{a+b} \text{ are in H.P.}$$

*Example* 3. The *n*th term of an A.P. is $\frac{n}{5}+2$, find the sum of 49 terms.

Let $a$ be the first term, and $l$ the last; then by putting $n=1$, and $n=49$ respectively, we obtain

$$a=\frac{1}{5}+2, \qquad l=\frac{49}{5}+2;$$

$$\therefore\ s=\frac{n}{2}(a+l)=\frac{49}{2}\left(\frac{50}{5}+4\right)$$

$$=\frac{49}{2}\times 14 = 343.$$

*Example* 4. If $a$, $b$, $c$, $d$, $e$ be in G.P. prove that $b+d$ is the geometric mean between $a+c$ and $c+e$.

Since $a$, $b$, $c$, $d$, $e$ are in continued proportion,

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e};$$

∴ each ratio $$=\frac{a+c}{b+d}=\frac{b+d}{c+e}, \qquad \text{[Art. 294.]}$$

Whence $$(b+d)^2=(a+c)(c+e).$$

## EXAMPLES XXXV.

1. Find the 6th term of the series $4, 2, 1\frac{1}{3}, \ldots$
2. Find the 21st term of the series $2\frac{1}{2}, 1\frac{12}{13}, 1\frac{9}{16}, \ldots$
3. Find the 8th term of the series $1\frac{1}{3}, 1\frac{11}{17}, 2\frac{2}{13}, \ldots$
4. Find the *n*th term of the series $3, 1\frac{1}{2}, 1, \ldots$

Find the series in which

5. The 15th term is $\frac{1}{25}$, and the 23rd term is $\frac{1}{41}$.

6. The 2nd term is 2, and the 31st term is $\frac{4}{31}$.

7. The 39th term is $\frac{1}{11}$, and the 54th term is $\frac{1}{26}$.

Find the harmonic mean between

8. 2 and 4. 9. 1 and 13. 10. $\frac{1}{4}$ and $\frac{1}{10}$.

11. $\frac{1}{a}$ and $\frac{1}{b}$. 12. $\frac{1}{x+y}$ and $\frac{1}{x-y}$. 13. $x+y$ and $x-y$.

14. Insert two harmonic means between 4 and 12.

15. Insert three harmonic means between $2\frac{2}{5}$ and 12.

16. Insert four harmonic means between 1 and 6.

17. If $G$ be the geometric mean between two quantities $A$ and $B$, shew that the ratio of the arithmetic and harmonic means of $A$ and $G$ is equal to the ratio of the arithmetic and harmonic means of $G$ and $B$.

18. To each of three consecutive terms of a G.P. the second of the three is added. Shew that the three resulting quantities are in H.P.

Sum the following series:

19. $1+1\frac{3}{4}+3\frac{1}{16}+\ldots\ldots$ to 6 terms.

20. $1+1\frac{3}{4}+2\frac{1}{2}+\ldots\ldots$ to 6 terms.

21. $(2a+x)+3a+(4a-x)+\ldots\ldots$ to $p$ terms.

22. $1\frac{4}{5}-1\frac{1}{5}+\frac{4}{5}-\ldots\ldots$ to 8 terms.

23. $1\frac{4}{5}+1\frac{1}{5}+\frac{3}{5}+\ldots\ldots$ to 12 terms.

24. If $x-a$, $y-a$, and $z-a$ be in G.P., prove that $2(y-a)$ is the harmonic mean between $y-x$ and $y-z$.

25. If $a$, $b$, $c$, $d$ be in A.P., $a$, $e$, $f$, $d$ in G.P., $a$, $g$, $h$, $d$ in H.P. respectively; prove that $ad=ef=bh=cg$.

26. If $a^2$, $b^2$, $c^2$ be in A.P., prove that $b+c$, $c+a$, $a+b$ are in H.P.

**27.** If $a$, $b$, $c$ be in A.P., and $\alpha$, $\beta$, $\gamma$ in H.P., shew that

$$\frac{a+c}{b\beta}=\frac{\alpha+\gamma}{a\gamma}.$$

**28.** If $a$ be the arithmetic mean between $b$ and $c$, and $b$ the geometric mean between $a$ and $c$, prove that $c$ will be the harmonic mean between $a$ and $b$.

**29.** If $\frac{a+b}{2}$, $b$, $\frac{b+c}{2}$ be in H.P., then $a$, $b$, $c$ are in G.P.

**30.** If $a$, $b$, $c$, $d$, $e$ be in G.P., prove that $c(a+2c+e)=(b+d)^2$.

**31.** If $a$, $b$, $c$, $d$... be a series of quantities in G.P., shew that the reciprocals of $a^2-b^2$, $b^2-c^2$, $c^2-d^2$,... are also in G.P.; and find the sum of $n$ terms of this latter series in terms of $a$ and $b$.

**32.** If $a$, $b$, $c$ be in A.P., and $b$, $c$, $d$ in H.P. then $a$, $\frac{c^2}{d}$, $c$ are in H.P., and $b$, $\frac{ad}{b}$, $d$ are also in H.P.

**33.** If $g$ be the geometric and $a$ the arithmetic mean between $m$ and $n$, and if $k^2$ be the arithmetic mean between $m^2$ and $n^2$, prove that $a^2$ is the arithmetic mean between $g^2$ and $k^2$.

**34.** If $a$, $b$, $c$, $d$ be in G.P., prove that $(b-c)^2=ac+bd-2ad$.

**35.** If $a$, $b$, $c$, $d$ be in G.P., prove that
$(a+d)(a-b)^2 : a(a-c)(a-d)=a-b+c : a+b+c.$

**36.** If $a$, $b$, $c$ be in H.P., prove that

$$\frac{1}{a}+\frac{1}{b+c},\ \frac{1}{b}+\frac{1}{c+a},\ \frac{1}{c}+\frac{1}{a+b}$$

are also in H.P.

## MISCELLANEOUS EXAMPLES.

1. Simplify $b-\{b-(a+b)-[b-(b-\overline{a-b})]+2a\}$.

2. Find the sum of
$$a+b-2(c+d),\ b+c-3(d+a) \text{ and } c+d-4(a+b).$$

3. Multiply $\frac{1}{2}x+\frac{2}{3}y$ by $x-\frac{1}{3}y$.

4. If $x=6$, $y=4$, $z=3$, find the value of $\sqrt[3]{2x+3y+z}$.

5. Find the square of $2-3x+x^2$.

6. Solve $\frac{x+3}{x-1}+\frac{x-4}{x-6}=2$.

7. Find the H.C.F. of $a^3-2a-4$ and $a^3-a^2-4$.

8. Simplify $\frac{2a}{a+b}+\frac{2b}{a-b}-\frac{a^2+b^2}{a^2-b^2}$.

9. Solve
$$\left.\begin{aligned}\frac{3}{5}x+\frac{y}{4}&=13\\ \frac{1}{3}x-\frac{y}{8}&=\ 3\end{aligned}\right\}.$$

10. Two digits, which form a number, change places when 18 is added to the number, and the sum of the two numbers thus formed is 44: find the digits.

11. If $a=1$, $b=-2$, $c=3$, $d=-4$, find the value of
$$\frac{a^2b^2+b^2c+d(a-b)}{10a-(c+b)^2}.$$

12. Subtract $-x^2+y^2-z^2$ from the sum of
$$\frac{1}{3}x^2+\frac{1}{4}y^2,\quad \frac{1}{5}y^2+\frac{1}{3}z^2,\quad \text{and } \frac{1}{3}z^2-\frac{1}{4}x^2$$

13. Write down the cube of $x+8y$.

14. Simplify $\frac{x^2+xy}{x^2+y^2}\times\frac{x^4-y^4}{xy+y^2}\times\frac{y}{x}$.

15. Solve $\frac{3}{5}(2x-7)-\frac{2}{3}(x-8)=\frac{4x+1}{15}+4$.

16. Find the H.C.F. and L.C.M. of
$$x^4+x^3+2x-4 \text{ and } x^3+3x^2-4.$$

**17.** Find the square root of $4a^4+9(1-2a)+3a^2(7-4a)$.

**18.** Solve
$$\left.\begin{aligned} y&=\frac{x+a}{2}+\frac{b}{3} \\ x&=\frac{y+b}{2}+\frac{a}{3} \end{aligned}\right\}.$$

**19.** Simplify $\left(\dfrac{a}{x+a}-\dfrac{x}{x-a}\right)\div\dfrac{x^2+a^2}{x^2+ax}$.

**20.** When 1 is added to the numerator and denominator of a certain fraction the result is equal to $\frac{3}{2}$; and when 1 is subtracted from its numerator and denominator the result is equal to 2: find the fraction.

**21.** Shew that the sum of $12a+6b-c$, $-7a-b+c$ and $a+b+6c$, is six times the sum of $25a+13b-8c$, $-13a-13b-c$, and $-11a+b+10c$.

**22.** Divide $x^2-xy+\frac{3}{16}y^2$ by $x-\frac{1}{4}y$.

**23.** Add together $18\left\{\dfrac{2x}{9}-\dfrac{1}{6}\left(\dfrac{2y}{3}+z\right)\right\}$,
$$24\left(\frac{3x}{8}-\frac{2y-3z}{12}\right), \text{ and } 30\left\{\frac{7z}{15}-\frac{4}{5}(2x-y)\right\}.$$

**24.** Find the factors of

(1) $10x^2+79x-8$. (2) $729x^6-y^6$.

**25.** Solve $\dfrac{2x-1}{5}+\dfrac{5x+3}{17}=3-\dfrac{4x-118}{11}$.

**26.** Find the value of
$$(5a-3b)(a-b)-b\{3a-c(4a-b)-b^2(a+c)\},$$
when $a=0$, $b=-1$, $c=\frac{1}{2}$.

**27.** Find the H.C.F. of
$$7x^3-10x^2-7x+10 \text{ and } 2x^3-x^2-2x+1.$$

**28.** Simplify $\dfrac{x^2-7xy+12y^2}{x^2+5xy+6y^2}\div\dfrac{x^2-5xy+4y^2}{x^2+xy-2y^2}$.

**29.** Solve
$$\left.\begin{aligned} 3abx+y&=9b \\ 4abx+3y&=17b \end{aligned}\right\}.$$

**30.** Find the two times between 7 and 8 o'clock when the hands of a watch are separated by 15 minutes.

**31.** If $a=1$, $b=-2$, $c=3$, $d=-4$, find the value of

$$\sqrt{d^2-4b+a^2}-\sqrt{c^3+b^3+a+d}.$$

**32.** Multiply the product of $\frac{1}{4}x^2-\frac{1}{2}xy+y^2$ and $\frac{1}{2}x+y$ by $x^3-8y^3$.

**33.** Simplify by removing brackets

$$a^4-\{4a^3-(6a^2-4a+1)\}$$
$$-[-2-\{a^4-(-4a^3-\overline{6a^2-4a})\}-(8a-1)].$$

**34.** Find the remainder when $5x^4-7x^3+3x^2-x+8$ is divided by $x-4$.

**35.** Simplify $\frac{x^2+y^2}{x^2-xy}\times\frac{xy-y^2}{x^4-y^4}\times\frac{x}{y}$.

**36.** Solve

$$\left.\begin{aligned}\frac{x-11}{3}+y&=18\\ 2x+\frac{y-13}{4}&=29\end{aligned}\right\}.$$

**37.** Find the square root of $4x^6-12x^4+28x^3+9x^2-42x+49$.

**38.** Solve $\cdot006x-\cdot491+\cdot723x=-\cdot005$.

**39.** Find the L.C.M. of $x^3+y^3$, $3x^2+2xy-y^2$ and $x^3-x^2y+xy^2$.

**40.** A bill of 25 guineas is paid with crowns and half-guineas, and twice the number of half-guineas exceeds three times that of the crowns by 17; how many of each are used?

**41.** Simplify

$$(a+b+c)^2-(a-b+c)^2+(a+b-c)^2-(-a+b+c)^2.$$

**42.** Find the remainder when $a^4-3a^3b+2a^2b^2-b^4$ is divided by $a^2-ab+2b^2$.

**43.** If $a=0$, $b=1$, $c=-2$, $d=3$, find the value of

$$(3abc-2bcd)\sqrt[3]{a^3bc-c^3bd+3}.$$

**44.** Find an expression which will divide both $4x^2+3x-10$ and $4x^3+7x^2-3x-15$ without remainder.

**45.** Simplify $\dfrac{a+\dfrac{ab}{a-b}}{a^2-\dfrac{2a^2b^2}{a^2+b^2}}\times\dfrac{\dfrac{1}{a^2}-\dfrac{1}{b^2}}{\dfrac{1}{a}-\dfrac{1}{b}}$.

**46.** Find the cube root of $8x^3-2x^2y+\frac{xy^2}{6}-\frac{y^3}{216}$.

**47.** Solve

$$\left.\begin{aligned}9x+8y&=43xy\\ 8x+9y&=42xy\end{aligned}\right\}.$$

**48.** Simplify $\dfrac{3}{x-4}-\dfrac{2}{x-5}-\dfrac{x-7}{(x-2)(x-3)}$.

**49.** Find the L.C.M. of $8x^3+38x^2+59x+30$
and $6x^3-13x^2-13x+30$.

**50.** A boy spent half of his money in one shop, one-third of the remainder in a second, and one-fifth of what he had left in a third. He had one shilling at last; how much had he at first?

**51.** Find the remainder when $x^7-10x^6+8x^5-7x^3+3x-11$ is divided by $x^2-5x+4$.

**52.** Simplify $4\left\{a-\dfrac{3}{2}\left(b-\dfrac{4c}{3}\right)\right\}\left\{\dfrac{1}{2}(2a-b)+2(b-c)\right\}$.

**53.** If $a=\dfrac{25}{16}$, $b=1$, $c=\dfrac{3}{4}$, prove that

$$(a-\sqrt{b})(\sqrt{a}+b)\sqrt{a-b}=\frac{3c^4}{\sqrt{a-c^2}}.$$

**54.** Find the L.C.M. of $x^2-7x+12$, $3x^2-6x-9$, and $2x^2-6x-8$.

**55.** Find the sum of the squares of $ax+by$, $bx-ay$, $ay+bx$, $by-ax$; and express the result in factors.

**56.** Solve $\dfrac{x}{6}+\dfrac{y}{4}=\dfrac{3x-5z}{4}=\dfrac{z}{8}+\dfrac{7y}{16}=1$.

**57.** Simplify $\dfrac{a^3+b^3}{a^4-b^4}-\dfrac{a+b}{a^2-b^2}-\dfrac{1}{2}\left\{\dfrac{a-b}{a^2+b^2}-\dfrac{1}{a-b}\right\}$.

**58.** Solve $x-\left(3x-\dfrac{2x+5}{10}\right)=\dfrac{1}{6}(2x+67)+\dfrac{5}{3}\left(1+\dfrac{x}{5}\right)$.

**59.** Add together the following fractions:

$$\frac{2}{x^2+xy+y^2},\quad \frac{-4x}{x^3-y^3},\quad \frac{x^2}{y^2(x-y)^2},\quad \frac{-x^2}{x^3y-y^4}.$$

**60.** A man agreed to work for 30 days, on condition that for every day's work he should receive 3*s.* 4*d.*, and that for every day's absence from work he should forfeit 1*s.* 6*d.*; at the end of the time he received £3. 11*s.*: how many days did he work?

**61.** Divide $\dfrac{3x^5}{4}+27-\dfrac{43x^2}{4}-4x^4+\dfrac{77x^3}{8}-\dfrac{33x}{4}$ by $\dfrac{x^2}{2}+3-x$.

**62.** Find the value of

$$\frac{4y}{5}(y-x)-35\left[\frac{3x-4y}{5}-\frac{1}{10}\left\{3x-\frac{5}{7}(7x-4y)\right\}\right]$$

when $x=-\dfrac{1}{2}$ and $y=2$.

**63.** Simplify $\dfrac{10x-11}{3(x^2-1)} - \dfrac{10x-1}{3(x^2+x+1)} + \dfrac{x^2-2x+5}{(x^3-1)(x+1)}$.

**64.** Find the cube root of $\dfrac{a^3c^3}{b^3}x^6 - \dfrac{3a^2c}{b}x^5 + \dfrac{3ab}{c}x^4 - \dfrac{b^3}{c^3}x^3$.

**65.** Solve $\dfrac{4x-17}{x-4} + \dfrac{10x-13}{2x-3} = \dfrac{8x-30}{2x-7} + \dfrac{5x-4}{x-1}$.

**66.** Find the factors of

(1) $x^3+5x^2+x+5$. (2) $x^2-2xy-323y^2$.

**67.** Solve

$$\left.\begin{aligned} \frac{1}{3}(x+y)+2z&=21 \\ 3x-\frac{1}{2}(y+z)&=65 \\ x+\frac{1}{2}(x+y-z)&=38 \end{aligned}\right\}.$$

**68.** Simplify $\dfrac{x+2y}{\frac{2}{7}x-y} - \dfrac{3x^2+63xy+70y^2}{2x^2+3xy-35y^2}$.

**69.** Find the square root of $-(3b-2c-2a)^3\{2(a+c)-3b\}$.

**70.** The united ages of a man and his wife are six times the united ages of their children. Two years ago their united ages were ten times the united ages of their children, and six years hence their united ages will be three times the united ages of the children. How many children have they?

**71.** Find the sum of $x^2-3xy-\dfrac{2}{3}y^2$, $2y^2-\dfrac{2}{3}y^3+z^2$, $xy-\dfrac{1}{3}y^2+y^3$, and $2xy-\dfrac{1}{3}y^3$.

**72.** From $\{(a+b)(a-x)-(a-b)(b-x)\}$ subtract $(a+b)^2-2bx$.

**73.** If $a=5$, $b=4$, $c=3$, find the value of

$$\sqrt[3]{6abc+(b+c)^3+(c+a)^3+(a+b)^3-(a+b+c)^3}.$$

**74.** Find the factors of

(1) $3x^3+6x^2-189x$. (2) $a^2+2ab+b^2+a+b$.

**75.** Solve

$$\left.\begin{aligned} px&=qy \\ (p+q)x-(q-p)y&=r \end{aligned}\right\}.$$

**76.** Simplify $\dfrac{x+\frac{y}{2}}{2x^2+xy+\frac{y^2}{2}} - \dfrac{x^2-\frac{y^2}{2}}{4\left(x^3-\frac{y^3}{8}\right)}$.

**77.** Solve $\dfrac{x-7}{x+7}+\dfrac{1}{2(x+7)}=\dfrac{2x-15}{2x-6}$.

**78.** Reduce $\dfrac{x^4-x^2-2x+2}{2x^3-x-1}$ to its lowest terms.

**79.** Add together the fractions:

$$\frac{1}{2x^2-4x+2},\quad \frac{1}{2x^2+4x+2},\quad \text{and}\quad \frac{1}{1-x^2}.$$

**80.** A number consists of three digits, the right-hand one being zero. If the left-hand and middle digits be interchanged the number is diminished by 180; if the left-hand digit be halved, and the middle and right-hand digit be interchanged, the number is diminished by 336: find the number.

**81.** Divide $1-5x+\dfrac{152}{15}x^3-\dfrac{106}{225}x^4-\dfrac{28}{9}x^5$ by $1-x-\dfrac{14}{15}x^2$.

**82.** If $p=1$, $q=\dfrac{1}{2}$, find the value of

$$\frac{(p^2+q^2)-(p-q)\sqrt{p^2+2pq+q^2}}{2p+q-\{p-(q-p)\}}.$$

**83.** Multiply $\dfrac{3x^3}{2}-5x^2+\dfrac{x}{4}+9$ by $\dfrac{x^2}{2}-x+3$.

**84.** Find the L. C. M. of

$$(a^2b-2ab^2)^2,\ 2a^2-3ab-2b^2,\ \text{and}\ 2(2a^2+ab)^2.$$

**85.** Solve $\dfrac{2x+3}{x+1}=\dfrac{4x+5}{4x+4}+\dfrac{3x+3}{3x+1}$.

**86.** Reduce $\dfrac{5x^3-14x^2+16}{3x^3-2x^2+16x-48}$ to its lowest terms.

**87.** Find the square root of

$$4a^4+9\left(a^2+\frac{1}{a^2}\right)+12a(a^2+1)+18.$$

**88.** Solve

$$\left.\begin{aligned}\frac{x}{2a}+\frac{y}{3b}&=a+b\\ \frac{3x}{a}-\frac{2y}{b}&=6(b-a)\end{aligned}\right\}.$$

**89.** Multiply

$$3x+4y+\frac{11xy}{x-\frac{3}{2}y}\ \text{by}\ 10x-3y-\frac{11xy}{\frac{x}{4}+y}.$$

**90.** A bag contained five pounds in shillings and half-crowns; after 17 shillings and 6 half-crowns were taken out, thrice as many half-crowns as shillings were left: find the number of each coin.

**91.** Find the value of

$$5(a-b)-2\{3a-(a+b)\}+7\{(a-2b)-(5a-2b)\},$$

when $a=-\frac{1}{9}b$.

**92.** Divide $3x^4-5x^3+7x^2-11x-13$ by $3x-2$.

**93.** Find the L.C.M. of

$$15(p^3+q^3),\ 5(p^2-pq+q^2),\ 4(p^2+pq+q^2) \text{ and } 6(p^2-q^2).$$

**94.** Resolve into factors:

(1) $a^3-8b^{15}$.  (2) $-x^2+2x-1+x^4$.

**95.** Solve $\dfrac{x+a}{x+b}=\dfrac{x+3a}{x+a+b}$.

**96.** Simplify

(1) $\dfrac{35a^2b^2c^2-49b^3c^3}{65a^5bc-91a^3b^2c^2}$.  (2) $\dfrac{y^4-7y^3+8y^2-12y}{2y^2-2y-60}$.

**97.** Solve

$$\left.\begin{aligned} 7x-9y+4z&=16\\ \frac{x+y}{3}&=\frac{x+y+z}{2}\\ 2x-3y+4z-5&=0 \end{aligned}\right\}.$$

**98.** Simplify $\dfrac{y^2-\dfrac{2y}{y-1}}{y^2-\dfrac{2y}{y+1}}\div\left(\dfrac{y^2-5y-6}{y^2-6y+5}\times\dfrac{y-2}{y+2}\right)$.

**99.** Find the square root of

$$\frac{4a^2-12ab-6bc+4ac+9b^2+c^2}{4a^2+9c^2-12ac}.$$

**100.** The express leaves Bristol at 3 p.m. and reaches London at 6; the ordinary train leaves London at 1.30 p.m. and arrives at Bristol at 6. If both trains travel uniformly, find the time when they will meet.

**101.** Solve (1) $\cdot\dot{6}x+\cdot75x-\cdot1\dot{6}=x-\cdot58\dot{3}x+5$.

(2) $\dfrac{37}{x^2-5x+6}+\dfrac{4}{x-2}=\dfrac{7}{3-x}$.

**102.** Simplify (1) $\dfrac{a+x}{a^2+ax+x^2}+\dfrac{a-x}{a^2-ax+x^2}+\dfrac{2x^3}{a^4+a^2x^2+x^4}$.

$$(2)\quad (1+x)^2 \div \left\{1+\cfrac{x}{1-x+\cfrac{x}{1+x+x^2}}\right\}.$$

**103.** Find the square root of

$$a^6+\frac{1}{a^6}-6\left(a^4+\frac{1}{a^4}\right)+15\left(a^2+\frac{1}{a^2}\right)-20;$$

also the cube root of the result.

**104.** Divide $1-2x$ by $1+3x$ to 4 terms.

**105.** I bought a horse and carriage for £75; I sold the horse at a gain of 5 per cent., and the carriage at a gain of 20 per cent., making on the whole a gain of 16 per cent. Find the original cost of the horse.

**106.** Find the divisor when $(4a^2+7ab+5b^2)^2$ is the dividend, $8(a+2b)^2$ the quotient, and $b^2(9a+11b)^2$ the remainder.

**107.** Solve (1) $5x(x-3)=2(x-7)$.

$$(2)\quad \frac{1}{(x-1)(x-2)}+6=\frac{3}{x-2}+\frac{2}{x-1}.$$

**108.** If $x=a+b+\dfrac{(a-b)^2}{4(a+b)}$, and $y=\dfrac{a+b}{4}+\dfrac{ab}{a+b}$,

prove that $(x-a)^2-(y-b)^2=b^2$.

**109.** Find the square root of

$$49x^4+\frac{1051x^2}{25}-\frac{14x^3}{5}-\frac{6x}{5}+9.$$

**110.** Solve $\dfrac{a+x}{a^2+ax+x^2}+\dfrac{a-x}{a^2-ax+x^2}=\dfrac{3a}{x(a^4+a^2x^2+x^4)}$.

**111.** Subtract $\dfrac{x+3}{x^2+x-12}$ from $\dfrac{x+4}{x^2-x-12}$,

and divide the difference by $1+\dfrac{2(x^2-12)}{x^2+7x+12}$.

**112.** Find the H.C.F. and L.C.M. of

$2x^2+(6a-10b)x-30ab$ and $3x^2-(9a+15b)x+45ab$.

**113.** Solve (1) $2cx^2-abx+2abd=4cdx$.

$$(2)\quad \frac{x}{2(x+3)}-2\tfrac{5}{24}=\frac{x^2}{x^2-9}-\frac{8x-1}{4(x-3)}.$$

**114.** If $a=1$, $b=2$, $c=3$, $d=4$, find the value of

$$\frac{a^b+b^c+c^d}{b^a+c^b+d^c+(a+b)(b+c)}+3\,(a^a+b^b+c^c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

**115.** I rode one-third of a journey at 10 miles an hour, one-third more at 9, and the rest at 8 miles an hour; if I had ridden half the journey at 10, and the other half at 8 miles per hour, I should have been half a minute longer on the way: what distance did I ride?

**116.** The product of two factors is $(3x+2y)^3-(2x+3y)^3$, and one of the factors is $x-y$; find the other factor.

**117.** If $a+b=1$, prove that $(a^2-b^2)^2=a^3+b^3-ab$.

**118.** Resolve into factors:

(1) $x^3+y^3+3xy\,(x+y)$. (2) $m^3-n^3-m\,(m^2-n^2)+n\,(m-n)^2$.

**119.** Solve (1) $\left.\begin{aligned} x^3-y^3&=28\\ x^2+xy+y^2&=7\end{aligned}\right\}$. (2) $\left.\begin{aligned} x^2-6xy+11y^2&=9\\ x-3y&=1\end{aligned}\right\}$.

**120.** Find the square root of

$$(a-b)^4-2\,(a^2+b^2)\,(a-b)^2+2\,(a^4+b^4).$$

**121.** Simplify the fractions

(1) $\dfrac{1}{a^2-\dfrac{a^3-1}{a+\dfrac{1}{a+1}}}$. (2) $\dfrac{\left(1+\dfrac{1}{x}\right)\times\left(1-\dfrac{1}{x}\right)^2}{x-\dfrac{1}{x}}$.

**122.** Find the H.C.F. of

$a^2b+b^2c-abc-ab^2$ and $ax^2+ab-a^2-bx^2$.

**123.** A constituency had two-thirds of its number Conservatives: in an election 25 refused to vote, and 60 went over to the Liberals; the voters were now equal. How many voters were there altogether?

**124.** Solve (1) $\dfrac{x^2}{a+b}+(a-b)=\dfrac{2ax}{a+b}$.

(2) $\dfrac{3}{x}+\dfrac{2}{y}=6\left(\dfrac{1}{y}-\dfrac{1}{2x}\right)=2$.

**125.** Simplify (1) $\left(1+\dfrac{y^2+z^2-x^2}{2yz}\right)\div\left(1-\dfrac{x^2+y^2-z^2}{2xy}\right)$.

(2) $\dfrac{(x+1)^3-(x-1)^3}{(x+1)^4-(x-1)^4}$.

**126.** Divide

$x^4+(a-1)\,x^3-(2a+1)\,x^2+(a^2+4a-5)\,x+3a+6$

by $x^2-3x+a+2$.

**127.** Resolve into factors:

(1) $x^2+5xy-24y^2+x-3y$. (2) $x^3-\frac{4}{x}$.

**128.** Find the square root of $p^2-3q$ to three terms.

**129.** Solve (1) $\frac{x-5}{x-6}-\frac{x-6}{x-7}=\frac{x-1}{x-2}-\frac{x-2}{x-3}$.

(2) $ax+1=by+1=ay+bx$.

**130.** Find the H.C.F. of $3x^2+(4a-2b)x-2ab+a^2$ and $x^3+(2a-b)x^2-(2ab-a^2)x-a^2b$.

**131.** Simplify

(1) $\frac{(x^a)^3}{x^{b+c}}\times\frac{(x^b)^3}{x^{c+a}}\times\frac{(x^c)^3}{x^{a+b}}$. (2) $x^{\frac{1}{2}}y^{\frac{1}{3}}\left(\frac{y^{\frac{1}{4}}}{x^{\frac{1}{3}}}\right)^2\div\frac{y^{-\frac{1}{4}}}{x^{\frac{1}{4}}}$.

**132.** At a cricket match the contractor provided dinner for 24 persons, and fixed the price so as to gain $12\frac{1}{2}$ per cent. upon his outlay. Three of the cricketers being absent, the remaining 21 paid the fixed price for their dinner, and the contractor lost 1*s.*: what was the charge for the dinner?

**133.** Prove that $x(y+2)+\frac{x}{y}+\frac{y}{x}$ is equal to $a$, if

$$x=\frac{y}{y+1} \text{ and } y=\frac{a-2}{2}.$$

**134.** Find the cube root of

$$x^3-12x^2+54x-112+\frac{108}{x}-\frac{48}{x^2}+\frac{8}{x^3}.$$

**135.** Find the H.C.F. and L.C.M. of

$$x^3+2ax^2+a^2x+2a^3 \text{ and } x^3-2ax^2+a^2x-2a^3.$$

**136.** Simplify

(1) $42\left\{\frac{4x-3y}{6}-\frac{3x-4y}{7}\right\}-56\left\{\frac{3x-2y}{7}-\frac{2x-3y}{8}\right\}$,

(2) $\frac{4b+a}{3b+a}+\frac{a-4b}{a-3b}+\frac{a^2-3b^2}{a^2-9b^2}$.

**137.** Resolve $4a^2(x^3+18ab^2)-(32a^5+9b^2x^3)$ into four factors.

**138.** Solve (1) $5\sqrt{3x}-1=\sqrt{75x-29}$.

(2) $\frac{xy}{x+y}=70$, $\frac{xz}{x+z}=84$, $\frac{yz}{y+z}=140$.

**139.** Shew that the difference between

$$\frac{x}{x-a}+\frac{x}{x-b}+\frac{x}{x-c} \quad \text{and} \quad \frac{a}{x-a}+\frac{b}{x-b}+\frac{c}{x-c}$$

is the same whatever value $x$ may have.

**140.** Multiply $x^{\frac{3}{2}}+2y^{\frac{3}{2}}+3z^{\frac{3}{2}}$ by $x^{\frac{3}{2}}-2y^{\frac{3}{2}}-3z^{\frac{3}{2}}$.

**141.** Walking 4½ miles an hour, I start 1½ hours after a friend whose pace is 3 miles an hour: how long shall I be in overtaking him?

**142.** Express in the simplest form

$$(1)\quad (8^{\frac{2}{3}}+4^{\frac{3}{2}})\times 16^{-\frac{3}{4}}. \qquad (2)\quad \frac{\left\{9^n \cdot 3^2\times\frac{1}{3^{-n}}\right\}-27^n}{3^{3n}\times 9}.$$

**143.** Find the square root of

$$\frac{x}{y}+\frac{y}{x}+3-2\sqrt{\frac{x}{y}}-2\sqrt{\frac{y}{x}}.$$

**144.** Simplify

$$(1)\quad \left(\frac{x}{x-1}-\frac{1}{x+1}\right)\cdot\frac{x^3-1}{x^6+1}\cdot\frac{(x-1)^2(x+1)^2+x^2}{x^4+x^2+1}.$$

$$(2)\quad \left\{\frac{a^4-y^4}{a^2-2ay+y^2}\div\frac{a^2+ay}{a-y}\right\}\times\left\{\frac{a^5-a^3y^2}{a^3+y^3}\div\frac{a^4-2a^3y+a^2y^2}{a^2-ay+y^2}\right\}.$$

**145.** Find the value of

$$(1)\quad \sqrt{8}+\sqrt{50}-\sqrt{18}+\sqrt{48}. \qquad (2)\quad \sqrt{35+14\sqrt{6}}.$$

**146.** Solve (1) $\dfrac{x-b}{x-a}-\dfrac{x-a}{x-b}=\dfrac{2(a-b)}{x-(a+b)}$.

$$(2)\quad \left.\begin{array}{r} 2x+3y=1\tfrac{1}{2} \\ 4x^2+9xy+9y^2=11 \end{array}\right\}.$$

**147.** Shew that

$$\frac{(a+b)^3-c^3}{(a+b)-c}+\frac{(b+c)^3-a^3}{b+c-a}+\frac{(c+a)^3-b^3}{c+a-b}.$$

is equal to $2(a+b+c)^2+a^2+b^2+c^2$.

**148.** Divide $a-x+4a^{\frac{1}{4}}x^{\frac{3}{4}}-4a^{\frac{1}{2}}x^{\frac{1}{2}}$

by $a^{\frac{1}{2}}+2a^{\frac{1}{4}}x^{\frac{1}{4}}-x^{\frac{1}{2}}$.

**149.** Find the square root of

$$(a-1)^4+2(a^4+1)-2(a^2+1)(a-1)^2.$$

**150.** How much are pears a gross when 120 more for a sovereign lowers the price 2*d.* a score?

**151.** Shew that if a number of two digits is six times the sum of its digits, the number formed by interchanging the digits is five times their sum.

**152.** Find the value of

$$\frac{1}{(a-b)(b-c)}-\frac{1}{(b-c)(a-c)}-\frac{1}{(c-a)(b-a)}.$$

**153.** Multiply

$$3+5x-\frac{12+41x+36x^2}{4+7x} \text{ by } 5-2x+\frac{26x-8x^2-14}{3-4x}.$$

**154.** If $x-\frac{1}{x}=1$, prove that $x^2+\frac{1}{x^2}=3$, and $x^3-\frac{1}{x^3}=4$.

**155.** Solve (1) $\frac{3x}{11}+\frac{23}{x+4}=\frac{1}{3}(x+5)$.

(2) $\left.\begin{aligned}2x^2-3y^2&=23\\2xy-3y^2&=\ \ 3\end{aligned}\right\}.$

**156.** Simplify

(1) $1\frac{3}{5}\sqrt{20}-3\sqrt{5}-\sqrt{\frac{1}{5}}$.  (2) $\frac{\sqrt{x}}{y^{-\frac{1}{3}}}\left(\frac{\sqrt[4]{y}}{x^{\frac{1}{6}}}\right)\div\frac{y^{-\frac{1}{4}}}{x^{\frac{1}{4}}}$.

**157.** Find the H.C.F. of $(p^2-1)x^2+(3p-1)x-p(p-1)$ and $p(p+1)x^2-(p^2-2p-1)x-(p-1)$.

**158.** Reduce to its simplest form

$$\frac{ax+\frac{a}{y}}{x-\frac{1}{y}}\times\frac{x^2+\frac{1}{y^2}}{bx^2-\frac{b}{y^2}}\times\frac{\frac{1}{5}(xy-1)^2}{\frac{1}{3}(x^4y^4-1)}.$$

**159.** Find the square root of

(1) $1-2^{2n+1}+4^{2n}$.  (2) $9^n-2.6^n+4^n$.

**160.** A clock gains 4 minutes a day. What time should it indicate at 6 o'clock in the morning, in order that it may be right at 7.15 P.M. on the same day?

**161.** If $x=2+\sqrt{2}$, find the value of $x^2+\frac{4}{x^2}$.

**162.** Solve

(1) $\dfrac{\sqrt{x+a}}{\sqrt{x-b}}=\dfrac{\sqrt{x-a}}{\sqrt{x}}$.  (2) $\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}=3.$

**163.** Simplify

$$\frac{a^2}{(b-a)(c-a)}+\frac{b^2}{(c-b)(a-b)}+\frac{c^2}{(a-c)(b-c)}.$$

**164.** Find the product of $\frac{1}{5}\sqrt{5}$, $\frac{1}{2}\sqrt[3]{2}$, $\sqrt[6]{80}$, $\sqrt[3]{5}$, and divide

$$\frac{8-4\sqrt{5}}{\sqrt{5}+1} \text{ by } \frac{3\sqrt{5}-7}{5+\sqrt{7}}.$$

**165.** Resolve $9x^6y^2-576y^2-4x^8+256x^2$ into six factors.

**166.** Simplify

(1) $\dfrac{1-\dfrac{a^2}{(x+a)^2}}{(x+a)(x-a)}\div\dfrac{x(x+2a)}{(x^2-a^2)(x+a)^2}$.

(2) $\dfrac{6x^2y^2}{m+n}\div\left[\dfrac{3(m-n)x}{7(r+s)}\div\left\{\dfrac{4(r-s)}{21xy^2}\div\dfrac{r^2-s^2}{4(m^2-n^2)}\right\}\right]$.

**167.** Simplify (1) $\left(a^{1+\frac{q}{p}}\right)^{\frac{p}{p+q}}\div\sqrt[p]{\dfrac{a^{2p}}{(a^{-1})^{-p}}}$.

(2) $\sqrt{14-\sqrt{132}}$.

**168.** Find the H.C.F. and L.C.M. of

$$20x^4+x^2-1,\ 25x^4+5x^3-x-1,\ 25x^4-10x^2+1.$$

**169.** Solve (1) $a+x+\sqrt{2ax+x^2}=b$.

(2) $x+9\frac{5}{8}+\dfrac{1}{\dfrac{x}{7}+\dfrac{11}{8}}=8.$

**170.** The price of photographs is raised 3*s.* per dozen, and customers consequently receive seven less than before for a guinea: what were the prices charged?

**171.** If $\left(a+\dfrac{1}{a}\right)^2=3$, prove that $a^3+\dfrac{1}{a^3}=0.$

**172.** Find the value of

$$\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}+\frac{4ab}{x^2-4b^2}, \text{ when } x=\frac{ab}{a+b}.$$

**173.** Reduce to fractions in their lowest terms

(1) $$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\div\left(\frac{x+y+z}{x^2+y^2+z^2-xy-yz-zx}-\frac{1}{x+y+z}\right)+1.$$

(2) $$\left(1-\frac{56}{x+4}+\frac{42}{x+3}\right)\left(1+\frac{56}{x-4}-\frac{42}{x-3}\right).$$

**174.** Express as a whole number

$$(27)^{\frac{2}{3}}+(16)^{\frac{3}{4}}-\frac{2}{(8)^{-\frac{2}{3}}}+\frac{\sqrt[5]{2}}{(4)^{-\frac{2}{5}}}.$$

**175.** Simplify

(1) $\dfrac{n}{1-x^n}+\dfrac{n}{1-x^{-n}}.$ (2) $\sqrt[4]{97-56\sqrt{3}}.$

**176.** Solve

(1) $$\frac{x-4a}{x-3a}+\frac{x-5a}{x-4a}=\frac{x+6a}{x-4a}+\frac{x+5a}{x-3a}.$$

(2) $$\left.\begin{aligned}3x^2+xy+3y^2&=8\tfrac{1}{4}\\8x^2-3xy+8y^2&=17\tfrac{3}{4}\end{aligned}\right\}.$$

**177.** Find the square root of $\dfrac{a^2x^2+2ab^2x^3+b^4x^4}{a^{2m}+2a^mx^n+x^{2n}}.$

**178.** Simplify

(1) $\dfrac{b}{\sqrt{a}}\times\sqrt[3]{ac}\times\dfrac{\sqrt[4]{c^3}}{\sqrt{b}}\times\dfrac{\sqrt{b^{-1}}}{a^{-\frac{1}{6}}}.$ (2) $\left\{\dfrac{(9^{n+\frac{1}{4}})\times\sqrt{3\,.\,3^n}}{3\sqrt{3^{-n}}}\right\}^{\frac{1}{n}}.$

**179.** A boat's crew can row 8 miles an hour in still water: what is the speed of a river's current if it take them 2 hours and 40 minutes to row 8 miles up and 8 miles down?

**180.** If $a=x^2-yz$, $b=y^2-zx$, $c=z^2-xy$, prove that
$$a^2-bc=x\,(ax+by+cz).$$

**181.** Find a quantity such that when it is subtracted from each of the quantities $a$, $b$, $c$, the remainders are in continued proportion.

**182.** Simplify

(1) $$\left(x+y-\frac{1}{x+y-\dfrac{xy}{x+y}}\right)\times\frac{x^3-y^3}{x^2-y^2}.$$

(2) $$\frac{2\,(7x-4)}{6x^2-7x+2}+\frac{x-10}{6x^2-x-2}-\frac{2\,(4x-1)}{4x^2-1}.$$

**183.** Find the sixth root of

$$729-2916x^2+4860x^4-4320x^6+2160x^8-576x^{10}+64x^{12}.$$

**184.** Simplify

(1) $$\frac{1}{x+\sqrt{x^2-1}}+\frac{1}{x-\sqrt{x^2-1}}.$$

(2) $$\sqrt[4]{16}+\sqrt[3]{81}-\sqrt[3]{-512}+\sqrt[3]{192}-7\sqrt[6]{9}.$$

**185.** Solve (1) $$\cfrac{5}{6-\cfrac{5}{6-\cfrac{5}{6-x}}}=x.$$

(2) $$\left.\begin{array}{l} x^2y^2+192=28xy \\ x+y=8 \end{array}\right\}.$$

**186.** Simplify

$$\frac{b-c}{a^2-(b-c)^2}+\frac{c-a}{b^2-(c-a)^2}+\frac{a-b}{c^2-(a-b)^2}.$$

**187.** Solve (1) $$x-15\tfrac{3}{4}+\frac{5}{x-15\frac{3}{4}}=6.$$

(2) $$2(x+y^{-1})=3(x^{-1}-y)=4.$$

**188.** If $xy=ab(a+b)$ and $x^2-xy+y^2=a^3+b^3$ prove that

$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{b}-\frac{y}{a}\right)=0.$$

**189.** Find the H.C.F. of

$$(2a^2-3a-2)x^2+(a^2+7a+2)x-a^2-2a$$

and $$(4a^2+4a+1)x^2-(4a^2+2a)x+a^2.$$

**190.** Multiply $$\sqrt{2x}+\sqrt{2(2x-1)}-\frac{1}{\sqrt{2x}}$$

by $$\frac{1}{\sqrt{2x}}+\sqrt{2(2x-1)}-\sqrt{2x}.$$

**191.** Divide $a^4b^2+b^4c^2+c^4a^2-a^2b^4-b^2c^4-c^2a^4$
by $a^2b+b^2c+c^2a-ab^2-bc^2-ca^2$.

**192.** Simplify

(1) $$\frac{7}{2(x+1)}-\frac{1}{6(x-1)}-\frac{10x-1}{3(x^2+x+1)}.$$

(2) $$\left\{\frac{\sqrt{x+a}}{\sqrt{x-a}}-\frac{\sqrt{x-a}}{\sqrt{x+a}}\right\}\times\frac{\sqrt{x^3-a^3}}{\sqrt{(x+a)^2-ax}}.$$

**193.** If $p$ be the difference between any quantity and its reciprocal, $q$ the difference between the square of the same quantity and the square of its reciprocal, shew that

$$p^2(p^2+4)=q^2.$$

**194.** A man started for a walk when the hands of his watch were coincident between three and four o'clock. When he finished, the hands were again coincident between five and six o'clock. What was the time when he started, and how long did he walk?

**195.** If $n$ be an integer, shew that $7^{2n+1}+1$ is always divisible by 8.

**196.** Simplify
$$\frac{\left(p+\frac{1}{q}\right)^p\left(p-\frac{1}{q}\right)^q}{\left(q+\frac{1}{p}\right)^p\left(q-\frac{1}{p}\right)^q}.$$

**197.** Find the value of

(1) $\dfrac{7+3\sqrt{5}}{7-3\sqrt{5}}+\dfrac{7-3\sqrt{5}}{7+3\sqrt{5}}$.

(2) $\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$ when $x=\dfrac{2b}{b^2+1}$.

**198.** If $a+b+c+d=2s$, prove that
$$4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2=16(s-a)(s-b)(s-c)(s-d).$$

**199.** A man buys a number of articles for £1, and sells for £1. 1*s.* all but two at 2*d.* apiece more than they cost: how many did he buy?

**200.** Find the square root of
$$2(81x^4+y^4)-2(9x^2+y^2)(3x-y)^2+(3x-y)^4.$$

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